

Infinitesimal thickenings of Morava K -theories

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Abstract

A. Baker has constructed certain sequences of cohomology theories which interpolate between the Johnson–Wilson and the Morava K -theories. We realize the representing sequences of spectra as sequences of MU -algebras. Starting with the fact that the spectra representing the Johnson–Wilson and the Morava K -theories admit such structures, we construct the sequences by inductively forming singular extensions. Our methods apply to other pairs of MU -algebras as well.

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1. Introduction

The Johnson–Wilson theories and the Morava K -theories constitute two families of multiplicative cohomology theories that play a central role in the approach to stable homotopy theory known as the chromatic point of view. It is an important fact that they can be represented by \mathbb{S} -algebras [5,27], meaning that the homotopy ring structures on suitably chosen representing spectra $E(n)$ and $K(n)$ (for a fixed prime p) can be rigidified in some point-set category of spectra, like the ones described in [13] or [20], to strict ones. Moreover, there are maps of \mathbb{S} -algebras $\eta_n: MU \rightarrow E(n)$ and $\rho_n: E(n) \rightarrow K(n)$, where MU is a chosen commutative \mathbb{S} -algebra representing complex cobordism. It can be arranged that both η_n and $\rho_n \eta_n$ are central maps, giving $E(n)$ and $K(n)$ the structure of MU -algebras [5,22].

One application, which is the starting point for this paper, is Baker’s construction [4] of a sequence of $E(n)$ -module spectra under $E(n)$ of the form

$$E(n) \longrightarrow \cdots \longrightarrow E(n)/I_n^{s+1} \longrightarrow E(n)/I_n^s \longrightarrow \cdots \longrightarrow E(n)/I_n = K(n). \quad (1.1)$$

Its homotopy limit is homotopy equivalent to $\widehat{E(n)}$, the Bousfield localization of $E(n)$ with respect to $K(n)$, and the induced sequence of homotopy groups is isomorphic to the canonical sequence of projections

$$E(n)_* \longrightarrow \cdots \longrightarrow E(n)_*/I_n^{s+1} \longrightarrow E(n)_*/I_n^s \longrightarrow \cdots \longrightarrow E(n)_*/I_n \cong K(n)_*,$$

where I_n denotes the kernel of the map induced by ρ_n on coefficient rings.

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The aim of this paper is to strengthen the bond between $E(n)$ and $K(n)$ that the sequence (1.1) provides, by constructing it as a sequence of MU -algebras. In fact, our construction applies to a variety of other pairs of MU -algebra spectra T and F , forming what we call a regular pair of MU -algebras (Definition 2.3). This notion embraces a natural class of MU -algebras T and F with commutative coefficient rings which are related by an MU -algebra map $T \rightarrow F$ that induces the projection of T_* onto a regular quotient $T_*/I \cong F_*$ on coefficients. Besides $(E(n), K(n))$, examples are the pairs $(BP, P(n))$, $(\widehat{E}(n), K(n))$, (E_n, K_n) and $(BP\langle n \rangle, k(n))$ for an arbitrary prime p . The role of the ground ring spectrum is not restricted to MU . It can be played by any even commutative \mathbb{S} -algebra R , meaning that its homotopy groups are trivial in odd degrees.

Theorem 1.1. *Let (T, F) be a regular pair over an even commutative \mathbb{S} -algebra R . There is a sequence of R -algebra spectra under T of the form*

$$T \longrightarrow \cdots \longrightarrow T/I^{s+1} \longrightarrow T/I^s \longrightarrow \cdots \longrightarrow T/I = F,$$

such that the induced sequence of coefficients is the canonical sequence of T_ -algebras*

$$T_* \longrightarrow \cdots \longrightarrow T_*/I^{s+1} \longrightarrow T_*/I^s \longrightarrow \cdots \longrightarrow T_*/I \cong F_*.$$

According to this result, the coefficients of T/I^s are isomorphic to the ring of functions on the scheme known as the infinitesimal thickening or neighbourhood of order $s - 1$ of the closed subscheme $\mathrm{spec}(T_*/I) \subseteq \mathrm{spec}(T_*)$ defined by I (see e.g. [16, II.9] or [21, 1.3]). We therefore refer to the R -algebras T/I^s as infinitesimal thickenings of $F = T/I$.

The proof of Theorem 1.1 relies on the results of [31]. There we construct such “topological I -adic towers” as towers of R -module spectra. This is done in such a way that I^s/I^{s+1} , the homotopy fibre of $T/I^{s+1} \rightarrow T/I^s$, is equivalent to a wedge of suspensions of F . In the case where $T = R$ and F is homotopy commutative, topological I -adic towers were first constructed in [6], using different methods.

As an application of Theorem 1.1, we obtain a qualitative statement concerning the relation of T and F if $T = R$. We briefly recall some definitions of [28]. A map of commutative R -algebras $A \rightarrow B$ is said to be symmetrically étale if the canonical map $B \rightarrow \mathrm{THH}^A(B)$ to the topological Hochschild homology of B relative to A is an equivalence and B is dualizable as an A -module. For an algebra D over a commutative \mathbb{S} -algebra C , the structure map $C \rightarrow D$ is defined to be symmetrically Henselian if each symmetrically étale map has the unique lifting property up to contractible choice with respect to $C \rightarrow D$. Now let F be a regular quotient algebra of an even commutative \mathbb{S} -algebra R , by which we mean that (R, F) is a regular pair of R -algebras. Consider the R -algebra $\widehat{R} = \mathrm{holim}_s R/I^s$. It may be interpreted as the Bousfield localization $L_F^R(R)$ of the R -module spectrum R with respect to F [6, 31]. Therefore, it admits the structure of a commutative R -algebra [13, Thm. VIII.2.2]. Rognes [28, Prop. 9.6.2] has proved the following statement for homotopy commutative F , using the so-called external I -adic tower constructed in [6, 22]. The sequence provided by Theorem 1.1 leads to a new proof, which is valid for any regular quotient algebra, not necessarily homotopy commutative.

Corollary 1.2. *Let F be a regular quotient algebra of an even commutative \mathbb{S} -algebra R . Then the natural map $\widehat{R} \rightarrow F$ is symmetrically Henselian.*

We construct the infinitesimal thickenings T/I^s of $F = T/I$ inductively as singular extensions. This method produces an R -algebra B from “strict derivations” $A \rightarrow A \vee M$, i.e. maps of R -algebras over A from a given R -algebra A to the square-zero extension $A \vee M$ of A by some A -bimodule spectrum M . Homotopy classes of such maps are possible to construct thanks to the fact that they correspond to homotopy classes of A -bimodule maps from D_A , the fibre of the multiplication map $A \wedge_R A \rightarrow A$, to M [22]. They are therefore strongly related to $\mathrm{THH}_R^*(A, M)$, the topological Hochschild cohomology groups of A with coefficients in M .

As I^s/I^{s+1} splits into a wedge of suspensions of F , the crucial calculation for the inductive step of the construction is the identification of $\mathrm{THH}_R^*(T/I^s, F)$. We will see that the problem can be reduced to the case where $T = R$. The situation is fundamentally different for $s = 1$ and $s > 1$. For $s = 1$, a certain universal coefficient spectral sequence converging to $\mathrm{THH}_R^*(F, F)$ collapses for degree reasons, but the extension problem is quite delicate. We show that R/I^2 can be constructed as a singular extension of F , irrespective of the specific R -algebra structure of F .

For $s > 1$, we first compute the endomorphism algebra $F_{R/I^s}^*(F)$ of F , viewed as a left R/I^s -module spectrum, by means of a Bousfield–Kan spectral sequence. After that, we construct a natural right coaction of $F_R^*(F)$, the coalgebra

of R -linear endomorphisms of F , on $F_{R/I^s}^*(F)$ and a natural map

$$\mathrm{THH}_R^*(R/I^s, F) \longrightarrow P(F_{R/I^s}^*(F))$$

into the coaction primitives of $F_{R/I^s}^*(F)$. We prove that it is an isomorphism, using the fact that $F_{R/I^s}^*(F)$ is relatively injective over $F_R^*(F)$.

Our approach shows that sequences of infinitesimal thickenings exist, but does not provide a way to classify all possible algebra structures on the R/I^s . For our constructions, we have to make choices of certain strict derivations. It would be desirable to know how different choices affect the obtained algebra structures. Essential to pursuing this question is a thorough understanding of $\mathrm{THH}_R^*(R/I^s, F)$ for $s \geq 1$. As already mentioned, this problem is very different for $s = 1$ and for $s > 1$. Recent results of Angeltveit [2] concerning the former may help to classify algebra structures on R/I^2 which are compatible with a given one on F . We compute $\mathrm{THH}_R^*(R/I^s, F)$ for $s > 1$ in this paper, but our identification is not natural and not canonical. We hope to obtain a natural description in future work, by using additional structure on $\mathrm{THH}_R^*(R/I^s, F)$.

The organization of the paper is as follows. In Section 2, we recall the results from [31] about the construction of topological I -adic towers. We add an argument that shows that the construction is independent of the choice of a regular sequence generating I . In Section 3, we recall the definition and basic facts about singular extensions of algebra spectra. In Section 4, we construct the first infinitesimal thickening of F . In Section 5, we relate $\mathrm{THH}_R^*(G, F)$ and $F_G^*(F)$, for an R -algebra G over a regular quotient algebra F of R , in the way indicated above. We characterize a situation which allows us to identify $F_G^*(F)$ and $\mathrm{THH}_R^*(G, F)$. In Section 6, we use these results to construct the higher infinitesimal thickenings of F inductively.

Notation and Conventions. We will use the results, notation and terminology from [13]. Throughout the paper, we work over a fixed even commutative \mathbb{S} -algebra R . We abbreviate \wedge_R by \wedge . Associated to an R -algebra A are the strict and the derived categories \mathcal{M}_A and \mathcal{D}_A of left A -module spectra, which have the same objects. If we make a statement about an “ A -module”, we have in mind an A -module spectrum viewed as an object in \mathcal{D}_A . A right A -module is a module over the opposite R -algebra A^{op} . If B is another R -algebra, an A - B -bimodule is by definition an $A \wedge B^{\mathrm{op}}$ -module. By an A -bimodule, we mean an A - A -bimodule. We write $b.\mathcal{M}_A$ and $b.\mathcal{D}_A$ for the strict and derived categories of A -bimodules. A free A -module is one equivalent to a wedge of suspensions of A . For two A -modules M and N , we write $\mathcal{D}_A^*(M, N)$ or $[M, N]_A^*$ for the graded R^* -module of graded morphisms from M to N in \mathcal{D}_A . It coincides with the homotopy groups $\pi_{-*}(F_A(M, N))$ of the internal function R -module $F_A(M, N)$. Here, as well as throughout the paper, our convention is that we mean the *derived* versions of functors like $F_A(-, -)$ whenever the arguments in question are specified as A -modules. If we want to consider the strict functors, we clearly declare the arguments as *strict* A -modules. If N is an A -module and M a right A -module, we write $N_A^*(-)$ for the cohomological functor $\mathcal{D}_A^*(-, N)$ and $M_*^A(-)$ for the homological functor $\pi_*(M \wedge_A -)$, both defined on \mathcal{D}_A . Recall that any object in the model categories \mathcal{M}_A is fibrant.

An R -ring spectrum is a homotopy R -algebra, i.e. a monoid in the homotopy category \mathcal{D}_R . The reduced homology and cohomology groups of an R -ring spectrum B with respect to an R -module F are defined as

$$\begin{aligned}\tilde{F}_*^R(B) &= \mathrm{coker}(F_* \cong F_*^R(R) \xrightarrow{F_*^R(\eta)} F_*^R(B)) \\ \tilde{F}_R^*(B) &= \ker(F_R^*(B) \xrightarrow{F_R^*(\eta)} F_R^*(R) \cong F^*).\end{aligned}$$

Here, as well as later on, we use the letter η to denote the unit of the R -algebra or R -ring spectrum in question. The product map will be denoted by μ .

We usually write 1 for identity maps. For clarity, we sometimes denote it by the same name as the object. An arrow labelled “can” denotes a canonically given map, which should be clear from the context. We follow the convention $X^* = X_{-*}$ for graded objects. If M_* and N_* are modules over a graded commutative ring Λ_* , we write $\mathrm{Hom}_{\Lambda_*}^*(M_*, N_*)$ and $\mathrm{End}_{\Lambda_*}^*(M_*)$ for the graded Λ_* -linear homomorphisms and endomorphisms, respectively. The upper-case stars refer to the cohomological grading of morphisms: An element $f \in \mathrm{Hom}^k(M_*, N_*)$ is a homomorphism of degree $-k$, i.e. $f_l: M_l \rightarrow N_{l-k}$. The tensor, exterior and symmetric algebras on M_* are denoted by $T_{\Lambda_*}^{*,*}(M_*)$, $\Lambda_{\Lambda_*}^{*,*}(M_*)$ and $\mathrm{Sym}_{\Lambda_*}^{*,*}(M_*)$, respectively. One grading comes from the construction of these algebras, the other one from M_* . We use the same symbols with just one upper-case star to denote the corresponding algebras,

graded by the total gradation. We write Mod_{Λ_*} for the category of graded modules over a graded ring Λ_* . An unadorned \otimes means the tensor product \otimes_{R_*} over the coefficients of the \mathbb{S} -algebra R .

2. Topological I -adic towers

Assume that $x = (x_1, \dots, x_n)$ is a given regular sequence of elements in the coefficient ring $R_* = \pi_*(R)$ of R , generating an ideal J . Then the homotopy groups of the R -module L_x , defined as

$$L_x = R/x_1 \wedge R/x_2 \wedge \cdots \wedge R/x_n,$$

are isomorphic to R_*/J . Here we use the notation R/x_i to denote the cofibre of x_i , viewed as a graded endomorphism of R in the homotopy category \mathcal{D}_R . If x' is another regular sequence generating the same ideal J , there is an equivalence $L_x \rightarrow L_{x'}$ of R -modules [13, Cor. V.2.10]. As R does not have any endomorphisms of odd degrees, the equivalence is canonical. So from now on, we will omit x from the notation and alternatively write R/J for L .

Under the additional hypothesis that each element x_i of a chosen sequence x is a non-zerodivisor, we can endow each R/x_i with an R -ring structure [30, Prop. 3.1]. Taken together, these products define one on L . Furthermore, it has been observed in [1] that the argument used in [27] to prove that the Morava K -theories admit A_∞ -structures can be generalized to show that in the present situation R/J can be promoted to an R -algebra.

Definition 2.1. We call $L = R/J$, endowed with a chosen R -ring structure, a *regular quotient ring* of R . If in addition an R -algebra structure on L is fixed, we refer to L as a *regular quotient algebra* of R .

Examples 2.2. The spectrum $\widehat{E(n)}$ representing completed Johnson–Wilson theory has a unique commutative \mathbb{S} -algebra structure [7] for all n and all primes p , so the Morava K -theory spectrum $K(n)$ can be realized as a regular quotient algebra of $\widehat{E(n)}$. Similarly, the Morava E -theory or Lubin–Tate spectrum E_n is a commutative \mathbb{S} -algebra in a unique way for any n and p [15,26]. The version of Morava K -theory whose representing spectrum is commonly denoted by K_n (see e.g. [14]) can then be constructed as a regular quotient algebra of E_n .

As mentioned in the Introduction, the setup just described is more restrictive than necessary and can be easily generalized to cover many more examples. Namely, assume that in addition to R and a regular sequence x of non-zerodivisors generating an ideal $J \triangleleft R_*$ as above, we are given an R -ring spectrum T whose coefficient ring T_* is commutative. Moreover, we require that x is regular on the R_* -module T_* as well. We write $I = J \cdot T_* \triangleleft T_*$. The R -module

$$F = T/I = T \wedge L = T \wedge R/x_1 \wedge \cdots \wedge R/x_n$$

has homotopy groups $F_* \cong T_* \otimes R_*/J \cong T_*/I$. An R -ring structure on L chosen as above and the given one on T induce an R -ring structure on F . Analogously, if T is given as an R -algebra, any R -algebra structure on L induces one on F .

Definition 2.3. If T is an R -ring spectrum satisfying the above conditions and an R -ring structure on L is fixed, we call $(T, F = T \wedge L)$ a *regular pair of R -ring spectra*. If moreover T is an R -algebra and an R -algebra structure on L is fixed, we say that (T, F) is a *regular pair of R -algebras*.

Remark 2.4. In [31], we considered regular pairs $(T, F = T \wedge L)$ of R -ring spectra with the additional property that the R -ring structure on T is commutative and called such triples (R, T, F) regular. An inspection shows that the construction of topological I -adic towers in [31] does not depend on the commutativity of T . The assumption that T_* is commutative is sufficient. In fact, T enters into the construction only at the end, as a certain Adams resolution of R with respect to L is smashed with T over R . Commutativity of T_* is used to identify the homotopy groups of the resulting tower.

Examples 2.5. The pairs $(E(n), K(n))$, $(BP, P(n))$, $(BP\langle n \rangle, k(n))$, $(\widehat{E(n)}, K(n))$ and (E_n, K_n) (see [14,25]) are all regular pairs of MU -algebras, for all n and p . For the first three pairs, this follows for instance from [1, Thm. 4.2] or is proved in [5]. For the last two pairs, it is a consequence of the fact that $E(n)$ is an MU -algebra, as the canonical map $E(n) \rightarrow \widehat{E(n)}$ is Bousfield localization with respect to $K(n)$ (see e.g. [31, Cor. 6.13]) and hence a

map of MU -algebras by [13, Thm. VIII.2.1], and because the canonical map $\widehat{E}(n) \rightarrow E_n$ is a map of commutative MU -algebras [28, Thm. 1.5].

In the following, $(T, F = T \wedge L)$ denotes a regular pair of R -ring spectra, with $F_* \cong T_*/I$ and $L_* \cong R_*/J$. The Bockstein operations $Q_1, \dots, Q_n \in L_R^*(L)$ are obtained by smashing

$$\rho_i \beta_i : R/x_i \longrightarrow \Sigma^{|x_i|+1} R/x_i$$

with the identities on the other smash factors of $L = R/x_1 \wedge \dots \wedge R/x_n$. Here β_i and ρ_i are taken from the cofibre sequences

$$\Sigma^{|x_i|} R \xrightarrow{x_i} R \xrightarrow{\rho_i} R/x_i \xrightarrow{\beta_i} \Sigma^{|x_i|+1} R.$$

The first part of the following result is [30, Prop. 4.15], and the second part is proved as [23, Lemma 2.6] or [1, Prop. 4.1].

Proposition 2.6. *There are isomorphisms of L_* -algebras*

$$L_R^*(L) \cong \Lambda_{L_*}(Q_1, \dots, Q_n), \quad L_*^R(L^{\text{op}}) \cong \Lambda_{L_*}(a_1, \dots, a_n),$$

with $|a_i| = |x_i| + 1$.

By construction, the Q_i depend on the choice of the sequence x . However, there is a “coordinate-free” description of $L_R^*(L)$, due to Strickland. We will use it to show that the construction of I -adic towers does not depend on the choice of x . Recall that a homotopy derivation $g : L \rightarrow M$ to an L -bimodule spectrum M is defined to be a map of R -modules making the diagram

$$\begin{array}{ccc} L \wedge L & \xrightarrow{1 \wedge g \vee g \wedge 1} & L \wedge M \vee M \wedge L \\ \downarrow & & \downarrow \\ L & \xrightarrow{g} & M \end{array}$$

commutative. The vertical maps are given by the product on L and the biaction of L on M . We write $\mathcal{D}er_R^k(L, M)$ for the set of derivations $L \rightarrow \Sigma^k M$. (We reserve the notation $\text{Der}_R^k(L, M)$ for strict derivations; see Section 3.) Strickland constructs a natural map $\mathcal{D}er_R^*(L, L) \rightarrow \text{Hom}_{R_*}^{*-1}(J/J^2, L_*)$ and shows that it is an isomorphism [30, Cor. 4.19]. The construction works for any L -bimodule M . Strickland’s result shows:

Proposition 2.7. *If M is L -free, there is a natural isomorphism*

$$\mathcal{D}er_R^*(L, M) \cong \text{Hom}_{R_*}^{*-1}(J/J^2, M_*). \quad (2.1)$$

The endomorphism algebra $L_R^*(L)$ is the exterior algebra generated by $\mathcal{D}er_R^*(L, L)$.

The construction of I -adic towers described in [31] can be slickened by using as an auxiliary object a certain R -algebra P , whose homotopy ring realizes a polynomial ring $R_*[y_1, \dots, y_n]$ in generators y_i of degree $|y_i| = |x_i|$. To define it, we need the tensor R -algebra construction $\mathbb{T}_R(M)$ on an R -module M [13, Thm. VII.2.9]:

$$\mathbb{T}_R(M) = R \vee M \vee (M \wedge M) \vee \dots$$

We let P be the R -algebra

$$P = \mathbb{T}_R(S_R^{|x_1|}) \wedge \dots \wedge \mathbb{T}_R(S_R^{|x_n|}),$$

where S_R^k is the cofibrant k -dimensional R -sphere. As a consequence of a Künneth isomorphism, the homotopy ring P_* is isomorphic to $R_*[y_1, \dots, y_n]$. Continuing the analogy between algebra and topology, we define $\text{gr}_I(T)$ to be $F \wedge P$ and $\text{gr}_J(R)$ to be $R/J \wedge P = L \wedge P$. The notation is legitimate, as the homotopy groups of $\text{gr}_I(T)$ realize the algebraic associated graded ring

$$\text{gr}_I(T_*) = T_*/I \oplus I/I^2 \oplus I^2/I^3 \oplus \dots$$

and the ones of $\mathrm{gr}_J(R)$ realize $\mathrm{gr}_J(R_*)$. The R -algebra P can be endowed with a grading, by assigning the spheres $S_R^{|x_i|}$ degree one. By construction, it is a graded free R -module spectrum. Furthermore, $\mathrm{gr}_I(T)$ and $\mathrm{gr}_J(R)$ are graded bimodules over P . So it makes sense to speak of their components of homogeneous degree s . We abuse notation and denote them by I^s/I^{s+1} and J^s/J^{s+1} respectively, as they have homotopy groups I^s/I^{s+1} and J^s/J^{s+1} .

Now let $\delta_0 = T \wedge \delta'_0: F \rightarrow \Sigma I/I^2$, where $\delta'_0: L \rightarrow \Sigma J/J^2$ corresponds to the identity of J/J^2 under the isomorphism (2.1) for $M = J/J^2$. Consider

$$F \xrightarrow{\delta_0} \Sigma I/I^2 \xrightarrow{\mathrm{can}} \Sigma \mathrm{gr}_I(T).$$

By extending scalars along the unit $R \rightarrow P$ of P , we obtain a map of graded P -modules $\delta_*: \mathrm{gr}_I(T) \rightarrow \Sigma \mathrm{gr}_I(T)$, with components

$$\delta_s: I^s/I^{s+1} \longrightarrow \Sigma I^{s+1}/I^{s+2}.$$

These maps determine a topological I -adic tower, by means of relative homological algebra in the triangulated category \mathcal{D}_R [12,24]. We briefly recall the terminology and refer to [31] for more details and additional references.

A relative injective resolution of an R -module M with respect to an R -ring spectrum E is a sequence of R -modules

$$* \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow I_2 \longrightarrow \cdots$$

with the following two properties. Firstly, each I_k is a retract of an R -module of the form $E \wedge X_k$ for some R -module X_k . Secondly, the image of the sequence under the functor $E \wedge -$ is split, i.e. equivalent to a sequence of the form

$$* \longrightarrow Z_1 \longrightarrow Z_1 \vee Z_2 \longrightarrow Z_2 \vee Z_3 \longrightarrow \cdots,$$

where the maps are the natural ones. The fundamental fact about relative injective resolutions is that each such determines a unique (up to non-canonical equivalence) diagram of the form (arrows with a circle denote maps of degree -1)

$$\begin{array}{ccccccc} M = M_0 & \xleftarrow{\circ} & M_1 & \xleftarrow{\circ} & M_2 & \xleftarrow{\circ} & M_3 & \cdots \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\ & I_0 & & I_1 & & I_2 & & \end{array}$$

which is composed of cofibre sequences

$$M_i \longrightarrow I_i \longrightarrow M_{i+1} \longrightarrow \Sigma M_i.$$

Let $i: I^{s+1} \rightarrow I^s$ and $p: I^s \rightarrow I^s/I^{s+1}$ denote the canonical inclusions and projections respectively.

Theorem 2.8 ([31, Thm. 6.6]). *The sequence of R -modules*

$$* \longrightarrow T \xrightarrow{\eta} F = T/I \xrightarrow{\delta_0} \Sigma I/I^2 \xrightarrow{\delta_1} \Sigma^2 I^2/I^3 \longrightarrow \cdots \quad (2.2)$$

is a relative injective resolution of T with respect to F . The homotopy groups of its associated Adams tower

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\iota} & I^3 & \xrightarrow{\iota} & I^2 & \xrightarrow{\iota} & I & \xrightarrow{\iota} & T \\ & \swarrow \varepsilon^3 & \downarrow \pi & \swarrow \varepsilon^2 & \downarrow \pi & \swarrow \varepsilon^1 & \downarrow \pi & \swarrow \varepsilon^0 & \downarrow \pi \\ & & I^3/I^4 & & I^2/I^3 & & I/I^2 & & T/I \end{array} \quad (2.3)$$

*realize the I -adic filtration of T_**

$$\begin{array}{ccccccc} \cdots & \xrightarrow{i} & I^3 & \xrightarrow{i} & I^2 & \xrightarrow{i} & I & \xrightarrow{i} & T_* \\ & & \downarrow p & & \downarrow p & & \downarrow p & & \downarrow p \\ & & I^3/I^4 & & I^2/I^3 & & I/I^2 & & T_*/I \end{array}$$

By “reversing” the tower in the statement of the theorem, using the octahedral axiom [19, Def. A.1.1], we obtain a tower

$$\begin{array}{ccccccc}
 T & \longrightarrow & \cdots & \xrightarrow{\rho} & T/I^3 & \xrightarrow{\rho} & T/I^2 \xrightarrow{\rho} T/I \\
 & & & & \uparrow & \swarrow \theta^2 & \uparrow & \swarrow \theta^1 \\
 & & & & I^2/I^3 & & I/I^2 &
 \end{array} \quad (2.4)$$

whose homotopy groups are isomorphic to what the notation suggests.

Proposition 2.9 ([31, Thm. 6.6]). *The long exact sequences of $F_R^*(-)$ -cohomology groups derived from (2.4) induce short exact sequence of free F^* -modules*

$$0 \longrightarrow \tilde{F}_R^*(T/I^{s+1}) \longrightarrow F_R^*(I^s/I^{s+1}) \longrightarrow \tilde{F}_R^{s+1}(T/I^s) \longrightarrow 0.$$

Remark 2.10. If T is an R -algebra, we have

$$F_T^*(F) \cong F_R^*(L) \cong \Lambda_{F^*}(Q_1, \dots, Q_n).$$

The proof of [31, Thm. 6.6] shows that there are short exact sequences of free F^* -modules

$$0 \longrightarrow \tilde{F}_T^*(T/I^{s+1}) \longrightarrow F_T^*(I^s/I^{s+1}) \longrightarrow \tilde{F}_T^{s+1}(T/I^s) \longrightarrow 0 \quad (2.5)$$

and that the sequences of Proposition 2.9 are isomorphic to the image of (2.5) under $F_R^*(T) \otimes -$.

Remark 2.11. The constructions described in this section can be generalized to cover quotients by ideals generated by infinite sequences. See [31].

Remark 2.12. An alternative construction of topological I -adic towers for the case where $T = R$ and where F is homotopy commutative is given in [6].

3. Singular extensions of \mathbb{S} -algebras

Let A be an R -algebra, not necessarily commutative. An R -algebra over A is an R -algebra B together with a map $\pi: B \rightarrow A$ of R -algebras. A map of R -algebras over A is a map of R -algebras which respects the structure maps. We write $\mathcal{A}_{R/A}$ for the resulting category of R -algebras over A . The model structure on the category of R -algebras gives rise to one on $\mathcal{A}_{R/A}$ [18, Prop. 1.1]. We denote its homotopy category by $\mathrm{Ho} \mathcal{A}_{R/A}$.

Associating to an A -bimodule M the square zero extension $A \vee M$ defines a functor $b\mathcal{M}_A \rightarrow \mathcal{A}_{R/A}$. Note that the inclusion $i: A \rightarrow A \vee M$ is a map in $\mathcal{A}_{R/A}$. If B is another R -algebra over A and $f: B \rightarrow A \vee M$ a map in $\mathrm{Ho} \mathcal{A}_{R/A}$, we can form the homotopy pullback of f and i in $\mathcal{A}_{R/A}$:

$$\begin{array}{ccc}
 C & \longrightarrow & B \\
 \downarrow & & \downarrow f \\
 A & \xrightarrow{i} & A \vee M
 \end{array} \quad (3.1)$$

Definition 3.1. The R -algebra C together with the map $C \rightarrow B$ from (3.1) is the *singular extension* associated to f .

By neglect of structure, (3.1) defines a homotopy pullback diagram in $b\mathcal{M}_C$ [11, 19.6]. Applying to this the forgetful functors $b\mathcal{M}_C \rightarrow \mathcal{M}_C \rightarrow \mathcal{M}_R$ gives homotopy pullback diagrams in \mathcal{M}_C and \mathcal{M}_R respectively. As $b\mathcal{D}_C$, \mathcal{D}_C and \mathcal{D}_R are all additive, C is equivalent to the homotopy fibre of the composition $\tilde{f}: B \xrightarrow{f} B \vee M \xrightarrow{\mathrm{can}} M$ in those categories. Thus, we obtain a cofibre sequence of C -bimodule spectra

$$C \longrightarrow B \xrightarrow{\tilde{f}} M \longrightarrow \Sigma M,$$

which maps to cofibre sequences in \mathcal{D}_C and \mathcal{D}_R under the forgetful functors.

The B -bimodule of associative differentials D_B is defined to be the homotopy fibre of the multiplication $B \wedge B \rightarrow B$. (We use the notation D_B rather than Ω_B from [22], to distinguish the notion from the module of commutative differentials of a commutative R -algebra, defined in [8].) Extending scalars along $B \rightarrow A$ yields the A -bimodule $D_B^A = A \wedge_B \tilde{D}_B \wedge_B A$, where \tilde{D}_B denotes a cofibrant replacement of D_B as a B -bimodule. According to our convention that smash products are to be taken in the derived sense, we would write this simply as $A \wedge_B D_B \wedge_B A$. We deviate from our convention here to avoid potential confusion. For an A -bimodule M , the R -module of derivations from B to M is defined as

$$\mathrm{Der}_R(B, M) = F_{A \wedge A^{\mathrm{op}}}(D_B^A, M).$$

We refer to the elements of

$$\mathrm{Der}_R^{-k}(B, M) = \pi_k(\mathrm{Der}_R(B, M)) = b\mathcal{D}_A^{-k}(D_B^A, M)$$

as strict derivations from B to M of degree k . This terminology is justified by the following result of Lazarev. The grading on the right-hand side is obtained by considering maps into the square zero extensions $A \vee \Sigma^k M$, for $k \in \mathbb{Z}$.

Theorem 3.2 ([22, Thm. 2.2]). *There is an adjunction*

$$\mathrm{Der}_R^*(B, M) \cong \mathrm{Ho} \mathcal{A}_{R/A}^*(B, A \vee M).$$

Recall that the topological Hochschild homology and cohomology groups of B with coefficients in a B -bimodule N are defined as

$$\mathrm{THH}_*^R(B, N) = \pi_*(\tilde{B} \wedge_{B \wedge B^{\mathrm{op}}} N), \quad \mathrm{THH}^*_R(B, N) = b\mathcal{D}_B^*(\tilde{B}, N),$$

respectively, where \tilde{B} denotes a cofibrant replacement of B as a B -bimodule. Regarding an A -bimodule M as a B -bimodule via restriction along $B \rightarrow A$, we obtain a long exact sequence of R^* -modules

$$\cdots \rightarrow \mathrm{THH}_R^*(B, M) \rightarrow M^* \rightarrow \mathrm{Der}_R^*(B, M) \rightarrow \mathrm{THH}_R^{*+1}(B, M) \rightarrow \cdots. \quad (3.2)$$

The “universal derivation” is defined to be the map of R -modules

$$d_B: B \xrightarrow{\eta_B} A \vee D_B^A \xrightarrow{\mathrm{can}} D_B^A,$$

where η_B is the unit of the adjunction. Via the forgetful functor $U: b\mathcal{D}_A \rightarrow \mathcal{D}_R$, d_B induces a “forgetful map”

$$V_B^*: \mathrm{Der}_R^*(B, M) \xrightarrow{U} \mathcal{D}_R^*(D_B^A, M) \xrightarrow{d_B^*} \mathcal{D}_R^*(B, M) = M_R^*(B).$$

If $\tilde{f}: B \rightarrow M$ is a given map of R -modules and $\varphi \in \mathrm{Der}_R^0(B, M)$ is chosen so that $V_B^*(\varphi) = \tilde{f}$, we say that “ φ realizes \tilde{f} as a strict derivation”.

By neglect of structure, the R -algebra map $f: B \rightarrow A \vee M$ over A corresponding to a strict derivation $\varphi: D_B^A \rightarrow M$ defines a map of R -ring spectra over A . By definition, this is nothing but a homotopy derivation in the sense of Section 2. This observation easily implies:

Lemma 3.3. *The image of the forgetful map $V_B^*: \mathrm{Der}_R^*(B, M) \rightarrow M_R^*(B)$ is contained in $\mathcal{D}_R^*(B, M)$. In particular, we always have $\mathrm{im}(V_B^*) \subseteq \tilde{M}_R^*(B)$.*

As pointed out in [22], there is an analogue of Theorem 3.2 for derivations over a not necessarily commutative R -algebra T . Instead of $\mathcal{A}_{R/A}$, we consider the category $\mathcal{A}_{T/A}$ of T -algebras over A . Its objects are R -algebras B with R -algebra maps $T \rightarrow B$ and $B \rightarrow A$. The morphisms are the R -algebra maps compatible with these structure maps. For similar reasons as before, $\mathcal{A}_{T/A}$ is a model category, whose homotopy category we denote by $\mathrm{Ho} \mathcal{A}_{T/A}$. The product map $B \wedge B \rightarrow B$ of a T -algebra B over A factors through $B \wedge_T B$. This makes B a monoid in the monoidal category of T -bimodules, with unit given by the structure map $T \rightarrow B$. The B -bimodule $D_{B/T}$ is defined as the homotopy fibre of $B \wedge_T B \rightarrow B$ and $D_{B/T}^A$ as $A \wedge_B \tilde{D}_{B/T} \wedge_B A$, with $\tilde{D}_{B/T}$ a cofibrant replacement of the B -bimodule $D_{B/T}$. The analogue to Theorem 3.2 is then [22, Rem. 2.4]

$$b\mathcal{D}_A^*(D_{B/T}^A, M) \cong \mathrm{Ho} \mathcal{A}_{T/A}^*(B, A \vee M).$$

We call the elements of $b\mathcal{D}_A^*(D_{B/T}^A, M)$ strict derivations from B to M over T .

4. Constructing the first-order infinitesimal thickening

From now on, we assume that $(T, F = T \wedge L)$ is a regular pair of R -algebras, with $F_* \cong T_*/I$, where $I = (x_1, \dots, x_n) \cdot T_*$ for some regular sequence (x_1, \dots, x_n) in R_* . The first aim of this section is to show that the homotopy derivation $\delta_0: F \rightarrow \Sigma I/I^2$ defined in Section 2 can always be realized as a strict derivation, independently of the chosen R -algebra structure on F . This has been shown by Lazarev for $R = T = MU$ and $F = H\mathbb{Z}/p$ [22, Thm. 10.2]. The proof for the general case is analogous. The argument given in [22] is rather brief, however, and it has taken the author some time to work out a complete proof. We therefore include it here, for the convenience of the reader.

Let us first assume that $R = T$, so $F = L$. As a consequence of Proposition 2.6, the E_2 -term of the universal coefficient spectral sequence [13, IV.4]

$$E_2^{*,*} = \text{Ext}_{F_*^R(F^{\text{op}})}^{*,*}(F^*, F^*) \implies \text{THH}_R^*(F, F) \quad (4.1)$$

is isomorphic to a polynomial algebra $F_*[z_1, \dots, z_n]$ in elements z_i of bidegree $(1, -|x_i| - 1)$. It will be more convenient to have a description which does not depend on the choice of a basis of $F_*^R(F^{\text{op}})$. Let us denote the augmentation ideal $\ker(\mu_*: F_*^R(F^{\text{op}}) \rightarrow F_*) \cong (D_F)_*$ by D and consider the short exact sequence of $F_*^R(F^{\text{op}})$ -modules

$$\mathcal{E}: 0 \longrightarrow D/D^2 \longrightarrow F_*^R(F^{\text{op}})/D^2 \longrightarrow F_* \longrightarrow 0.$$

Note that we may naturally identify

$$(D/D^2)^\vee := \text{Hom}_{F_*}(D/D^2, F_*) \cong \text{Hom}_{F_*^R(F^{\text{op}})}(D/D^2, F_*).$$

Thus, the connecting homomorphism associated to \mathcal{E} defines a map

$$(D/D^2)^\vee \longrightarrow \text{Ext}_{F_*^R(F^{\text{op}})}^{1,*}(F_*, F_*). \quad (4.2)$$

It induces an algebra isomorphism

$$\text{Sym}_{F_*}^{*,*}((D/D^2)^\vee) \cong \text{Ext}_{F_*^R(F^{\text{op}})}^{*,*}(F_*, F_*) \quad (4.3)$$

of the sort we were looking for. Now it is clear that the spectral sequence (4.1) collapses for degree reasons. Hence, we find that the associated graded of $\text{THH}_R^*(F, F)$, with respect to the spectral sequence filtration, is given by

$$\text{gr}^*(\text{THH}_R^*(F, F)) \cong \text{Sym}_{F_*}^*((D/D^2)^\vee).$$

Using the identification

$$\text{Der}_R^*(F, \Sigma I/I^2) \cong \ker(\text{THH}_R^*(F, \Sigma I/I^2) \longrightarrow (\Sigma I/I^2)^*), \quad (4.4)$$

and noting that $D/D^2 \cong (\Sigma I/I^2)_*$, we then have

$$\text{gr}^*(\text{Der}_R^*(F, \Sigma I/I^2)) \cong \widetilde{\text{Sym}}_{F_*}^*(\text{End}_{F^*}(D/D^2)), \quad (4.5)$$

where $\widetilde{\text{Sym}}_{F_*}^*(\text{End}_{F^*}(D/D^2))$ is the augmentation ideal. The solution of the extension problem depends on the R -algebra structure chosen on F [2]. For our purposes, it suffices to know $\text{Der}_R^*(F, \Sigma I/I^2)$ up to associated graded.

Lemma 4.1. *The sequence of homotopy groups of the cofibre sequence associated to the universal derivation $d_F: F \rightarrow D_F$ is of the form*

$$0 \longrightarrow D_F^* \cong D/D^2 \oplus D^2 \xrightarrow{i \oplus 1} R^{*+1}/I^2 \oplus D^2 \longrightarrow F^{*+1} \longrightarrow 0, \quad (4.6)$$

where i is the canonical inclusion $I/I^2 \rightarrow R^*/I^2$, with degree shifted by one.

Proof. By definition, d_F is the image of the identity on D_F under the forgetful map

$$b\mathcal{D}_F(D_F, D_F) \cong \text{Der}_R^*(F, D_F) \xrightarrow{V_F^*} \mathcal{D}_R^*(F, D_F).$$

It is therefore a homotopy derivation, by Lemma 3.3. Now consider the universal coefficient spectral sequence

$$E_2^{*,*} = \text{Ext}_{R^*}^{*,*}(F^*, D_F^*) \implies \mathcal{D}_R^*(F, D_F). \quad (4.7)$$

It was noted above that D_F is equivalent to a free F -module spectrum. Hence, as the analogous spectral sequence converging to $F_R^*(F)$ collapses, so does (4.7). It now suffices to show that the element representing d_F lies in the first Ext-group and corresponds to the short exact sequence (4.6) (see e.g. [31, Prop. 6.3]).

As $F_*^R(F)$ is F_* -free, the composition

$$\mathcal{D}_R^*(F, D_F) \longrightarrow \text{Hom}_{F_*}^*(F_*^R(F), F_*^R(D_F)) \longrightarrow \text{Hom}_{F_*}^*(F_*^R(F), (D_F)_*) \quad (4.8)$$

induced by applying first $F_*^R(-)$ and then using the left action of F on D_F , is an isomorphism (see e.g. [31, Lemma 6.2]). Now consider the following diagram, where the top map is (4.8), the left map is the natural inclusion, the right map is induced by the projection $F_*^R(F) \rightarrow D/D^2$ and the lower map is the isomorphism (2.1) from Proposition 2.7:

$$\begin{array}{ccc} \mathcal{D}_R^*(F, D_F) & \xrightarrow{\cong} & \text{Hom}_{F_*}^*(F_*^R(F), (D_F)_*) \\ \uparrow & & \uparrow \\ \mathcal{D}er_R^*(F, D_F) & \xrightarrow{\cong} & \text{Hom}_{F_*}^*(D/D^2, (D_F)_*). \end{array} \quad (4.9)$$

We leave it to the reader to verify that the diagram commutes. Alternatively, he/she may check that, for an F -free F -bimodule M , the restriction of the natural isomorphism [31, Lemma 6.2]

$$\mathcal{D}_R^*(F, M) \cong \text{Hom}_{F_*}^*(F_*^R(F), M_*)$$

to $\mathcal{D}er_R^*(F, M)$ factors through $\text{Hom}_{F_*}^*(D/D^2, M_*)$, and that the induced map is an isomorphism. This observation leads to another construction of the isomorphism (2.1).

We claim that the homomorphism $F_*^R(F) \rightarrow (D_F)_*$ corresponding to d_F , viewed as an R -module map, is given by the projection

$$F_*^R(F) \cong (D_F)_* \oplus F_* \longrightarrow (D_F)_*.$$

To show this, we use the fact [22, Thm. 4.1] that the composition $F \xrightarrow{d_F} D_F \xrightarrow{\text{can}} F \wedge F$ coincides with $\eta \wedge F - F \wedge \eta$. This map corresponds under the isomorphism

$$\mathcal{D}_R^*(F, F \wedge F) \cong \text{Hom}_{F_*}^*(F_*^R(F), F_*^R(F))$$

to the homomorphism $x \mapsto x - \mu_*(x) \cdot 1$. By naturality, this proves our claim. As diagram (4.9) commutes, it follows that the homotopy derivation d_F corresponds to the inclusion $D/D^2 \rightarrow (D_F)_*$. Now the short exact sequence

$$0 \longrightarrow D/D^2 \longrightarrow R^{*+1}/I^2 \longrightarrow F^{*+1} \longrightarrow 0$$

gives rise to a natural isomorphism

$$\gamma: \text{Hom}_{F_*}^*(D/D^2, (D_F)_*) \cong \text{Ext}_{R_*}^{1,*}(F_{*-1}, (D_F)_*).$$

It is not difficult to see that the induced isomorphism

$$\mathcal{D}er_R^*(F, D_F) \cong \text{Ext}_{R_*}^{1,*}(F_{*-1}, (D_F)_*)$$

maps a given derivation to the element which represents it in the spectral sequence (4.7). As γ sends the inclusion $D/D^2 \rightarrow (D_F)_*$ to (4.6), the lemma is proved. \square

Proposition 4.2. Any element in $\text{Der}_R^*(F, \Sigma I/I^2)$ lying in the coset corresponding under (4.5) to the identity of D/D^2 realizes $\delta_0: F \rightarrow \Sigma I/I^2$ as a strict derivation.

Proof. We have to check that the forgetful map V_F^* , defined as the composition

$$V_F^*: \operatorname{Der}_R^*(F, \Sigma I/I^2) \xrightarrow{U} \mathcal{D}_R^*(D_F, \Sigma I/I^2) \xrightarrow{d_F^*} \mathcal{D}_R^*(F, \Sigma I/I^2),$$

maps an element in the stated coset to δ_0 . To that end, we consider the morphisms induced by U and d_F^* on the universal coefficient spectral sequences

$$\begin{array}{ccc} (E_2^{*,*})_1 & = \operatorname{Ext}_{F_{-*}^R(F^{\operatorname{op}})}^{*,*}(D_F^*, (\Sigma I/I^2)^*) & \Longrightarrow \operatorname{Der}_R^*(F, \Sigma I/I^2) \\ & \downarrow U^* & \downarrow U \\ (E_2^{*,*})_2 & = \operatorname{Ext}_{R^*}^{*,*}(D_F^*, (\Sigma I/I^2)^*) & \Longrightarrow \mathcal{D}_R^*(D_F, \Sigma I/I^2) \\ & \downarrow d_F^* & \downarrow d_F^* \\ (E_2^{*,*})_3 & = \operatorname{Ext}_{R^*}^{*,*}(F^*, (\Sigma I/I^2)^*) & \Longrightarrow \mathcal{D}_R^*(F, \Sigma I/I^2). \end{array}$$

All three spectral sequences collapse. We have seen this for the first one above; for the second and the third one this is a consequence of [31, Prop. 6.5]. Here we use the fact that D_F , as an F -module spectrum with F^* -free homotopy groups, is equivalent to a free F -module spectrum.

By Lemma 4.1 below, $d_F^*: F^* \rightarrow D_F^*$ is trivial. Hence the induced morphism on the associated graded

$$\operatorname{gr}(d_F^*): \operatorname{gr}^*(\mathcal{D}_R^*(D_F, \Sigma I/I^2)) \longrightarrow \operatorname{gr}^*(\mathcal{D}_R^*(F, \Sigma I/I^2))$$

is zero and so gives rise to a map

$$\tilde{\operatorname{gr}}^*(d_F^*): \operatorname{gr}^*(\mathcal{D}_R^*(D_F, \Sigma I/I^2)) \longrightarrow \operatorname{gr}^{*+1}(\mathcal{D}_R^*(F, \Sigma I/I^2)).$$

We now check that the composite

$$\tilde{\operatorname{gr}}^*(d_F^*) \circ \operatorname{gr}^*(U): \operatorname{gr}^*(\operatorname{Der}_R^*(F, \Sigma I/I^2)) \longrightarrow \operatorname{gr}^{*+1}(\mathcal{D}_R^*(F, \Sigma I/I^2))$$

maps the coset x corresponding to the identity of D/D^2 to the coset representing δ_0 . The element of $\operatorname{Ext}_{F_{-*}^R(F^{\operatorname{op}})}^*(D_F^*, D/D^2)$ corresponding under

$$\operatorname{Ext}_{F_{-*}^R(F^{\operatorname{op}})}^*(D_F^*, D/D^2) \cong \operatorname{gr}^*(\operatorname{Der}_R^*(F, \Sigma I/I^2))$$

to x is the projection $q: D_F^* \cong D \rightarrow D/D^2$. By the Geometric Boundary Theorem (see [31, Prop. 6.4] for instance) and Lemma 4.1 below, $\tilde{\operatorname{gr}}^*(d_F^*)$ is represented by the connecting homomorphism of $\operatorname{Ext}_{R^*}^*(-, F^*)$ associated to the short exact sequence (4.6). It maps q to the short exact sequence

$$0 \longrightarrow D/D^2 \longrightarrow R^{*+1}/I^2 \longrightarrow F^{*+1} \longrightarrow 0,$$

which, by [31, Prop. 6.5], represents δ_0 . So we have shown that any element in the coset of x maps to δ_0 , modulo elements of filtration 2. However, the image of V_F is contained in the homotopy derivations, by Lemma 3.3. By Proposition 2.7, there are no homotopy derivations of filtration higher than one. Therefore the proposition is proved. \square

Corollary 4.3. *For any regular triple (R, T, F) , there is a realization*

$$\vartheta_0: D_{F/T} \rightarrow \Sigma I/I^2$$

of $\delta_0: F = T/I \rightarrow \Sigma I/I^2$ as a strict derivation over T .

Proof. We have proved this for $R = T$. For the general case, Proposition 4.2, applied to the regular pair of R -algebras (R, L) , guarantees the existence of a strict realization $\vartheta'_0: D_L \rightarrow \Sigma J/J^2$ of $\delta'_0: L \rightarrow \Sigma J/J^2$ from Section 2. Smashing it with the identity on T produces an F -bilinear map $T \wedge D_L \rightarrow \Sigma I/I^2$. But $D_{F/T}$, as the fibre of $\mu: F \wedge_T F \rightarrow F$, is equivalent to $T \wedge D_L$, and hence we are done. \square

Corollary 4.4. *The map $\rho: T/I^2 \rightarrow T/I$ from the tower (2.4) can be realized as a map of R -algebras under T .*

Proof. Form the singular extension associated to ϑ_0 . \square

Remark 4.5. Note that T/I^2 is equivalent as an R -algebra to the smash product of T with the singular extension on the derivation ϑ'_0 from the proof of [Corollary 4.3](#).

As a consequence of [Corollary 4.4](#), we obtain a cofibre sequence of T/I^2 -bimodules of the form

$$I/I^2 \longrightarrow T/I^2 \longrightarrow F \xrightarrow{\theta_0} \Sigma I/I^2. \quad (4.10)$$

A chosen element $\varphi \in (D/D^2)^\vee$ determines a map

$$\alpha_*: \mathcal{D}_{T/I^2}^*(F, \Sigma I/I^2) \cong \mathcal{D}_{T/I^2}^*(F, F) \otimes_{F^*} D/D^2 \xrightarrow{1 \otimes \varphi} \mathcal{D}_{T/I^2}^*(F, F).$$

This induces an F^* -linear map

$$(D/D^2)^\vee \longrightarrow F_{T/I^2}^*(F), \quad \varphi \longmapsto \alpha_*(\theta_0). \quad (4.11)$$

The composition with the natural map $F_{T/I^2}^*(F) \rightarrow F_R^*(F)$ corresponds to the inclusion of the homotopy derivations $\mathcal{D}er_R^*(F, F)$. In particular, (4.11) is injective. Let $\widehat{T}_{F^*}(M^*)$ denote the completed tensor algebra on an F^* -module M^* .

Proposition 4.6. *The homomorphism (4.11) extends to an algebra isomorphism*

$$\widehat{T}_{F^*}((D/D^2)^\vee) \cong F_{T/I^2}^*(F). \quad (4.12)$$

Proof. Consider the universal coefficient spectral sequence

$$E_2^{*,*} = \text{Ext}_{T^*/I^2}^{*,*}(F^*, F^*) \implies \mathcal{D}_{T/I^2}^*(F, F).$$

By [Lemma 4.8](#) below, the E_2 -term is isomorphic as an algebra to the tensor algebra $T_{F^*}^*((D/D^2)^\vee)$. By [31, Prop. 6.5] and the remarks above, the image of (4.11) is represented by the first Ext-group. Hence this consists of permanent cycles. As the spectral sequence is multiplicative, this forces it to collapse and hence to converge strongly. It follows that (4.12) induces an isomorphism on the associated graded modules. As the spectral sequence converges strongly, (4.12) itself is an isomorphism [10, Thm. 2.6]. \square

Corollary 4.7. *The restriction map $F_{T/I^2}^*(F) \rightarrow F_T^*(F)$ is surjective.*

Proof. It is a map of algebras and hence given by the projection

$$\widehat{T}_{F^*}((D/D^2)^\vee) \longrightarrow \Lambda_{F^*}^*((D/D^2)^\vee)$$

(compare [Remark 2.10](#)). \square

Lemma 4.8. *There is an isomorphism of algebras*

$$\text{Ext}_{R^*/I^2}^{*,*}(F^*, F^*) \cong T_{F^*}^*((D/D^2)^\vee).$$

Proof. It is easy to write down an R^*/I^2 -free resolution of F^* . Namely, let M^* be a free R^*/I^2 -module on generators y_i of (cohomological) degree $-|x_i|$. There is an R^*/I^2 -free resolution of F^* of the form

$$\dots \xrightarrow{\delta_4} M^* \otimes M^* \otimes M^* \xrightarrow{\delta_3} M^* \otimes M^* \xrightarrow{\delta_2} M^* \xrightarrow{\delta_1} R^*/I^2 \xrightarrow{\text{can}} F^*, \quad (4.13)$$

where $\otimes = \otimes_{R^*/I^2}$. The differential δ_s is defined as

$$\delta_s(u_1 \otimes \dots \otimes u_s) = \bar{\varphi}(u_1) \cdot u_2 \otimes \dots \otimes u_s,$$

where $\bar{\varphi}: M^* \rightarrow R^*/I^2$ evaluates y_i at x_i , so $\bar{\varphi}(y_i) = x_i$. We briefly explain where this complex comes from. Let $\mathcal{R}^* = R^*[y_1, \dots, y_n]$ be the polynomial algebra on generators y_i of degree as specified above. Let \mathcal{I} be the ideal $(y_1, \dots, y_n) \triangleleft \mathcal{R}^*$. View R^* as an \mathcal{R}^* -algebra via the homomorphism of algebras $\varphi: \mathcal{R}^* \rightarrow R^*$ defined by $\varphi(y_i) = x_i$. Let \mathcal{F}^* and \mathcal{G}^* be the \mathcal{R}^* -algebras $\mathcal{R}^*/\mathcal{I}$ and $\mathcal{R}^*/\mathcal{I}^2$ respectively. Note that as an R^* -module, \mathcal{F}^* is just

R^* . As $\varphi(\mathcal{I}) \subseteq I$, φ induces compatible homomorphisms $\mathcal{F}^* \rightarrow F^*$ and $\mathcal{G}^* \rightarrow G^* = R^*/I^2$, which in turn induce maps

$$R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{F}^* \rightarrow F^*, \quad R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{G}^* \rightarrow G^* \quad (4.14)$$

in the homotopy category of differential graded R^* -algebras [29, Section 5]. Here we use the standard notation $\otimes_{\mathcal{R}^*}^{\mathbf{L}}$ for the left derived functor of $\otimes_{\mathcal{R}^*}$. The first map of (4.14) is an equivalence, as

$$R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{F}^* \simeq \mathbb{K}_{\mathcal{R}^*}(y_1, \dots, y_n) \otimes_{\mathcal{R}^*} \mathcal{F}^* \simeq \mathbb{K}_{R^*}(x_1, \dots, x_n) \simeq F^*,$$

where \mathbb{K} denote Koszul complexes. Because $\mathcal{I}/\mathcal{I}^2$ is isomorphic to a sum of suspensions of \mathcal{F}^* , the natural map $R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{I}/\mathcal{I}^2 \rightarrow I/I^2$ is an equivalence as well. The diagram of cofibre sequences

$$\begin{array}{ccccccc} R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{I}/\mathcal{I}^2 & \longrightarrow & R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{G}^* & \longrightarrow & R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{F}^* & \longrightarrow & R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \Sigma \mathcal{I}/\mathcal{I}^2 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ I/I^2 & \longrightarrow & G^* & \longrightarrow & F^* & \longrightarrow & \Sigma I/I^2, \end{array}$$

where all the maps are the natural ones, and the 5-Lemma imply now that the second map of (4.14) is also an equivalence.

Using the equivalences (4.14), we derive an equivalence of homotopy differential graded algebras

$$G^* \otimes_{\mathcal{G}^*}^{\mathbf{L}} \mathcal{F}^* \simeq (R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{G}^*) \otimes_{\mathcal{G}^*}^{\mathbf{L}} \mathcal{F}^* \simeq R^* \otimes_{\mathcal{R}^*}^{\mathbf{L}} \mathcal{F}^* \simeq F^*. \quad (4.15)$$

This allows us to realize F^* as the bar construction $B^{R^*}(G^*, \mathcal{G}^*, \mathcal{F}^*)$ (this is analogous to [13, Prop. IX.2.3]). The latter is homotopy equivalent to the realization of the reduced bar resolution, which is of the form (with $\otimes = \otimes_{R^*}$)

$$\cdots \rightrightarrows G^* \otimes \mathcal{I}/\mathcal{I}^2 \otimes \mathcal{I}/\mathcal{I}^2 \otimes \mathcal{F}^* \rightrightarrows G^* \otimes \mathcal{I}/\mathcal{I}^2 \otimes \mathcal{F}^* \rightrightarrows G^* \otimes \mathcal{F}^*.$$

The chain complex associated to this simplicial resolution is precisely the chain complex (4.13) from before.

Mapping (4.13) into F^* kills all the differentials, so the statement is true additively. One way to account for the multiplicative structure is by using a duality argument. One checks that the resolution (4.13) is compatible with the standard comultiplication on $T_{R^*}^*(M^*)$ and uses the fact that the multiplication on $\mathrm{Ext}_{G^*}^{*,*}(F^*, F^*)$ is dual to the comultiplication on $\mathrm{Tor}_{G^*}^{G^*}(F_*, F_*)$. Another possibility is to use the fact that the cosimplicial R^* -module obtained by applying $\mathrm{Hom}_{G^*}^*(-, F^*)$ to the resolution $B_{G^*}^{R^*}(G^*, \mathcal{G}^*, \mathcal{F}^*)$ admits a composition product which is compatible with the one on Ext -groups. This is explained in more detail in the proof of a later result, Theorem 5.12, in an analogous situation. \square

5. Topological Hochschild cohomology and ordinary cohomology

Let F be a regular quotient algebra of R and G an R -algebra over F , with structure map $\pi: G \rightarrow F$. We can view F as a G -bimodule via pullback along π and consider $\mathrm{THH}_R^*(G, F)$. Forgetting the right G -action, we can also ask about the left G -linear endomorphisms $F_G^*(F)$ of F . The aim of this section is to relate these two invariants and to characterize a situation where we can determine both of these. In a first part, we establish a connection between bimodule cohomology $F_{F \wedge F^{\mathrm{op}}}^*(-)$ and (left) module cohomology $F_F^*(-)$. In a second part, we apply the results to compare $\mathrm{THH}_R^*(G, F)$ and $F_G^*(F)$ and to use this to compute the two groups in a special situation.

5.1. Bimodule cohomology

Let F be a regular quotient algebra of R and let $F^e = F \wedge F^{\mathrm{op}}$ denote its “enveloping algebra”. To compare $F_{F^e}^*(M)$ and $F_F^*(M)$ for an F -bimodule M , we are going to use the fact that $F_F^*(M)$ is a right comodule over the coalgebra $F_R^*(F)$. The comultiplication on $F_R^*(F)$ is induced by the product of F in the usual way. We have to be a bit careful in defining it, however, as we do not insist on F being homotopy commutative. For this reason, we give some details.

Proposition 5.1. *The graded endomorphisms $F_R^*(F)$ admit a natural augmented F^* -coalgebra structure.*

Proof. The first thing to notice is that a priori several F^* -actions on $F_R^*(F)$ have to be distinguished. There are two natural right and two natural left F^* -actions, defined in an obvious way. Fortunately, they all agree in the present situation. The reason is that the four R^* -actions obtained via pulling back along the surjection $\eta_*: R_* \rightarrow F_*$ necessarily coincide, because R is commutative. We define the comultiplication as the composition

$$\Delta: F_R^*(F) \xrightarrow{\mu^*} F_R^*(F \wedge F) \cong F_R^*(F) \otimes_{F^*} F_R^*(F)$$

of the map induced by the product μ on F and a version of the Künneth isomorphism [31, Prop. 2.6] valid for non-commutative ring spectra. The counit is induced by the unit η , by means of $\eta^*: F_R^*(F) \rightarrow F_R^*(R) \cong F^*$. The augmentation is defined as $\nu^*: F^* \cong F_F^*(F) \rightarrow F_R^*(F)$, via “restriction along η ”. It is a coalgebra map because $\Delta(\text{id}_F) = \text{id}_F \otimes \text{id}_F$. \square

Let $U: \mathcal{D}_{F^e} \rightarrow \mathcal{D}_F$ denote the “restriction functor” along the R -algebra map $1 \wedge \eta: F \cong F \wedge R \rightarrow F \wedge F$. Let $\text{Cohom}_{F_R^*(F)}^*(-, -)$ denote right $F_R^*(F)$ -colinear maps and $P(-) = \text{Cohom}_{F_R^*(F)}^*(F^*, -)$ the functor of primitives.

Proposition 5.2. *The functor $F_F^*(U(-))$ takes values in right $F_R^*(F)$ -comodules. There is a natural transformation*

$$\psi: F_{F^e}^*(-) \rightarrow P(F_F^*(U(-))). \quad (5.1)$$

Proof. For the first part, we define a natural map for left F -modules X

$$F_F^*(X) \otimes_{F^*} F_R^*(F) \longrightarrow F_F^*(X \wedge F) \quad (5.2)$$

precisely analogous to the Künneth transformation cited above, by sending an element $f \otimes \varphi$ to the left F -linear map

$$X \wedge F \xrightarrow{f \wedge \varphi} F \wedge F \xrightarrow{\mu} F.$$

As (5.2) is an isomorphism on the suspensions of F , it is a natural equivalence, by the usual comparison argument for cohomology theories. If M is now an F -bimodule, we compose the map $F_F^*(M) \rightarrow F_F^*(M \wedge F)$ induced by the right action of F on M with the inverse of (5.2) for $X = U(M)$. It is straightforward to check that the obtained natural map

$$F_F^*(U(M)) \longrightarrow F_F^*(U(M)) \otimes_{F^*} F_R^*(F)$$

defines a coaction of $F_R^*(F)$ on $F_F^*(U(M))$.

To define a natural transformation of the stated form, we associate to an element $f \in F_{F^e}^*(M) = [M, F]_{F^e}^*$ the induced homomorphism of $F_R^*(F)$ -comodules

$$F^* \cong F_F^*(U(F)) \xrightarrow{F_F^*(U(f))} F_F^*(U(M)).$$

This yields a natural map of the required form. \square

Let C^* be a graded coalgebra over a graded commutative ring Λ^* . Recall that a graded right comodule M^* over C^* is called relatively injective if the functor $\text{Cohom}_{C^*}^*(-, M^*)$ takes split short exact sequences of Λ^* -modules to short exact sequences (of Λ^* -modules). It is well known that M^* is relatively injective if and only if M^* is isomorphic to a retract of an extended C^* -comodule, i.e. a comodule of the form $X^* \otimes_{\Lambda^*} C^*$ for some Λ^* -module X^* (see e.g. [17, Lemma 3.1.2]). Recall that we may compute the derived functors $\text{Coext}_{C^*}^{*,*}(M^*, N^*)$ using relatively injective resolutions of M^* if M^* is Λ^* -projective [25, Lemmas A1.1.6 (b), A1.2.9(b)].

Proposition 5.3. *There are Adams-type spectral sequences*

$$\text{Coext}_{F_R^*(F)}^{*,*}(F^*, F_F^*(X)) \implies F_{F^e}^*(X),$$

whose edge homomorphism coincides with ψ_X from (5.1).

Proof. For given X , form the tower of F -bimodule spectra

$$\begin{array}{ccccccc} X = X_0 & \longrightarrow & X_1 & \longrightarrow & X_2 & \longrightarrow & \cdots \\ & \nwarrow & \swarrow & \nwarrow & \swarrow & \nwarrow & \\ & X_0 \wedge F & & X_1 \wedge F & & \cdots & \end{array}$$

where X_{s+1} is inductively defined as the cofibre of right scalar multiplication on X_s , $X_s \wedge F \rightarrow X_s$. Applying $F_F^*(U(-))$ gives a relative injective resolution of $F_F^*(U(X))$ over $F_R^*(F)$. Furthermore, for X of the form $X = Y \wedge F$, we have $F_{F^e}^*(X) \cong F_F^*(Y)$, and this isomorphism is induced by ψ_X from (5.1):

$$F_{F^e}^*(X) \rightarrow \operatorname{Hom}_{F_R^*(F)}^*(F^*, F_F^*(U(X))) \cong F_F^*(Y).$$

Therefore the spectral sequence exists by standard arguments. \square

Corollary 5.4. *If M is an F -bimodule for which $F_F^*(U(M))$ is a relatively injective $F_R^*(F)$ -comodule, we have a natural identification $F_{F^e}^*(M) \cong P(F_F^*(U(M)))$.*

Remark 5.5. In a completely analogous manner, we can compare $F_{F^e}^*(-)$ with $F_{F^{\text{op}}}^*(-)$. We then use the fact that $F_{F^{\text{op}}}^*(U'(M))$ has a natural left $F_R^*(F)$ -coaction for an F -bimodule M , where $U': \mathcal{D}_{F^e} \rightarrow \mathcal{D}_{F^{\text{op}}}$ is the obvious “restriction functor”.

Remark 5.6. There is a dual version for homology of the previous statements. The situation is considerably easier in homology, and we content ourselves with a brief sketch. Consider the algebra $F_*^e = \pi_*(F \wedge F^{\text{op}})$. The map $\mu_*: F_*^e \rightarrow F_*$ induced by the product map is an algebra map, as a consequence of the fact that it is F_*^e -linear. Hence F_*^e is an augmented F_* -algebra. Let $Q(M_*)$ denote the indecomposables $Q(M_*) = F_* \otimes_{F_*^e} M_*$ of a left F_*^e -module M_* . The homotopy groups M_* of an F -bimodule M admit an obvious left F_*^e -action. The analogue of the natural transformation (5.1) is now just given by the Künneth map

$$Q(M_*) = F_* \otimes_{F_*^e} M_* \longrightarrow F_*^{F^e}(M). \quad (5.3)$$

The existence of Künneth spectral sequences [13, Theorem IV.4.1] implies that (5.3) is an isomorphism whenever M_* is relatively projective over F_*^e .

5.2. Topological Hochschild cohomology of algebras over regular quotients

We recall the following fundamental fact:

Proposition 5.7 ([13, Prop. III.4.1]). *Let $\varphi: R \rightarrow R'$ be a map of \mathbb{S} -algebras. By pullback along φ , we obtain a functor $\varphi^*: \mathcal{M}_{R'} \rightarrow \mathcal{M}_R$. It has a left adjoint $\varphi_*: \mathcal{M}_R \rightarrow \mathcal{M}_{R'}$, given by $\varphi_*(M) = R' \wedge_R M$. The adjunction passes to one on derived categories.*

Recall the homotopy category $\operatorname{Ho} \mathcal{A}_{R/A}$ of R -algebras over a fixed R -algebra A from Section 3.

Proposition 5.8. *Let F be a regular quotient algebra of R .*

- (i) *The functor $F_\gamma^*(F): \operatorname{Ho} \mathcal{A}_{R/F} \rightarrow \operatorname{Mod}_{F^*}$ takes values in right $F_R^*(F)$ -comodules.*
- (ii) *There is natural transformation $\theta: \operatorname{THH}_R^*(-, F) \rightarrow P(F_\gamma^*(F))$. It is an isomorphism for those R -algebras G over F for which $F_G^*(F)$ is relatively injective over $F_R^*(F)$.*

Proof. Let G be an R -algebra over F with structure map $\pi: G \rightarrow F$.

(i) The adjunction of Proposition 5.7 obtained from π allows us to identify $F_G^*(F)$ with $F_F^*(F \wedge_G F)$. Now $F \wedge_G F$ is an F -bimodule, and hence $F_F^*(F \wedge_G F)$ admits a right $F_R^*(F)$ -coaction by Proposition 5.2.

(ii) The algebra map $\pi \wedge \pi: G^e \rightarrow F^e$ determines an adjunction between G^e - and F^e -modules, from which we deduce an isomorphism

$$\operatorname{THH}_R^*(G, F) = F_{G^e}^*(G) \cong F_{F^e}^*(F \wedge_G G \wedge_G F) \cong F_{F^e}^*(F \wedge_G F).$$

We recall our standing convention that all functors are taken in the derived sense when we work in homotopy categories. So we implicitly replace the G -bimodule G cofibrantly above. Now define θ_G as the composition

$$\operatorname{THH}_R^*(G, F) \cong F_{F^e}^*(F \wedge_G F) \xrightarrow{\psi_{F \wedge_G F}} P(F_F^*(F \wedge_G F)) \cong P(F_G^*(F)),$$

where $\psi_{F \wedge_G F}$ is the natural map from (5.1). The last statement is a consequence of Corollary 5.4. \square

To obtain information about $\mathrm{THH}_R^*(G, F)$, we are thus naturally led to consider $F_G^*(F)$. We will prove the following fact together with [Theorem 5.12](#) below.

Proposition 5.9. *The natural algebra map $p: F_G^*(F) \rightarrow F_R^*(F)$ is surjective.*

We define the augmentation ideal $I(F_G^*(F)) \subseteq F_G^*(F)$ to be the kernel of p :

$$I(F_G^*(F)) = \ker(p: F_G^*(F) \rightarrow F_R^*(F)).$$

The map p is $F_R^*(F)$ -colinear, if we regard the coalgebra $F_R^*(F)$ as a right comodule over itself. This is clear by construction of the right coaction of $F_R^*(F)$ on $F_G^*(F)$ and the comultiplication on $F_R^*(F)$. By left exactness of the primitives functor, we deduce a short exact sequence of F^* -modules

$$0 \longrightarrow P(I(F_G^*(F))) \longrightarrow P(F_G^*(F)) \longrightarrow F^* \longrightarrow 0. \quad (5.4)$$

On the other hand, we have a short exact sequence

$$0 \longrightarrow \mathrm{Der}_R^{*-1}(G, F) \longrightarrow \mathrm{THH}_R^*(G, F) \longrightarrow F^* \longrightarrow 0 \quad (5.5)$$

involving $\mathrm{THH}_R^*(G, F)$. It is obtained from the long exact sequence (3.2) for $A = M = F$ and $B = G$, by noting that $\mathrm{THH}_R^*(G, F) \rightarrow F^*$ sends π to 1 and is therefore surjective. The map $\theta_G: \mathrm{THH}_R^*(G, F) \rightarrow P(F_G^*(F))$ from [Proposition 5.8](#) is compatible with the surjections of (5.4) and (5.5) by construction. Hence it induces a map

$$\tilde{\theta}_G: \mathrm{Der}_R^{*-1}(G, F) \longrightarrow P(I(F_G^*(F))). \quad (5.6)$$

Recall the forgetful map $V_G^*: \mathrm{Der}_R^*(G, F) \rightarrow \tilde{F}_R^*(G)$ from Section 3.

Proposition 5.10. *There is a natural map of $F_R^*(F)$ -comodules*

$$W_G^*: I(F_G^*(F)) \longrightarrow \tilde{F}_R^{*-1}(G), \quad (5.7)$$

which makes the diagram

$$\begin{array}{ccc} \mathrm{Der}_R^{*-1}(G, F) & \xrightarrow{\tilde{\theta}_G} & P(I(F_G^*(F))) \\ & \searrow V_G^* & \swarrow W_G^*|_{P(I(F_G^*(F)))} \\ & \tilde{F}_R^{*-1}(G) & \end{array}$$

commutative.

We will prove this together with [Theorem 5.12](#) as well.

Proposition 5.11. *The topological Hochschild cohomology $\mathrm{THH}_R^*(B, A)$ of an R -algebra B over A is an R^* -algebra augmented over A^* , in a natural way.*

Proof. Let $\pi: B \rightarrow A$ be the structure map of B . Applying [Proposition 5.7](#) to the algebra map $1 \wedge \pi^{\mathrm{op}}: B \wedge B^{\mathrm{op}} \rightarrow B \wedge A^{\mathrm{op}}$, we obtain

$$\mathrm{THH}_R^*(B, A) = A_{B^e}^*(B) \cong A_{B \wedge A^{\mathrm{op}}}^*((B \wedge A^{\mathrm{op}}) \wedge_{B^e} B). \quad (5.8)$$

We claim that $(B \wedge A^{\mathrm{op}}) \wedge_{B^e} B \simeq A$ as $B \wedge A^{\mathrm{op}}$ -modules. To prove this, we use the bar resolution $B_*^R(\tilde{B}, \tilde{B}, \tilde{B})$ on a cofibrant replacement \tilde{B} of the R -module B (see [13, IX.2]). This is a simplicial strict \tilde{B}^e -module whose component of degree q is $\tilde{B} \wedge (\tilde{B})^{\wedge q} \wedge \tilde{B}$. The canonical map $\tilde{B} \wedge \tilde{B} \rightarrow \tilde{B} \wedge_{\tilde{B}} \tilde{B}$ defines an augmentation and induces a derived equivalence on applying geometric realization:

$$B^R(\tilde{B}, \tilde{B}, \tilde{B}) = |B_*^R(\tilde{B}, \tilde{B}, \tilde{B})| \simeq B \wedge_B B \simeq B.$$

Now geometric realization commutes with smash products, and so we find

$$\begin{aligned} (B \wedge A^{\text{op}}) \wedge_{B^e} B &\simeq (B \wedge A^{\text{op}}) \wedge_{B^e} |B_*^R(\tilde{B}, \tilde{B}, \tilde{B})| \\ &\simeq |(\tilde{B} \wedge \tilde{A}^{\text{op}}) \wedge_{\tilde{B}^e} B_*^R(\tilde{B}, \tilde{B}, \tilde{B})| \simeq |B_*^R(\tilde{B}, \tilde{B}, \tilde{A})| \simeq B \wedge_B A \simeq A, \end{aligned}$$

where \tilde{A} is a cofibrant replacement of the R -module A . Combining this equivalence with (5.8), we obtain

$$\text{THH}_R^*(B, A) \cong A_{B \wedge A^{\text{op}}}^*(A).$$

Via this identification, the composition product on $A_{B \wedge A^{\text{op}}}^*(A)$ defines an R^* -algebra structure on $\text{THH}_R^*(B, A)$. The augmentation is induced by the natural map of algebras $A_{B \wedge A^{\text{op}}}^*(A) \rightarrow A_{A^{\text{op}}}^*(A) \cong A^*$, obtained by pulling back along $\eta_B \wedge 1: A^{\text{op}} \cong R \wedge A^{\text{op}} \rightarrow B \wedge A^{\text{op}}$. \square

Here is the main result of this section:

Theorem 5.12. *Let F be a regular quotient algebra of R and let G be an R -algebra over F . Assume that $W_G^*: I(F_G^*(F)) \rightarrow \tilde{F}_R^{*-1}(G)$ is surjective and that $F_R^*(G)$ is F^* -free.*

(i) *There is an isomorphism of $F_R^*(F)$ -comodules*

$$F_G^*(F) \cong \hat{T}_{F^*}(\tilde{F}_R^{*-1}(G)) \otimes_{F^*} F_R^*(F).$$

(ii) *The forgetful map $V_G^*: \text{Der}_R^*(G, F) \rightarrow \tilde{F}_R^*(G)$ is split surjective. Any chosen section induces an isomorphism of algebras*

$$\hat{T}_{F^*}(\tilde{F}_R^{*-1}(G)) \cong \text{THH}_R^*(G, F).$$

Proof. (i) In the following, we assume that F and G are both cofibrant as R -modules, replacing them cofibrantly if necessary. Consider the two-sided bar resolution $B_*^R(G, G, G)$ of G . It is of the form

$$\cdots \rightrightarrows G \wedge G \wedge G \rightrightarrows G \wedge G \longrightarrow G. \quad (5.9)$$

On applying $F_G(- \wedge_G F, F)$, we obtain a cosimplicial R -module equivalent to one of the form

$$F_G(F, F) \longrightarrow F_R(F, F) \rightrightarrows F_R(G \wedge F, F) \rightrightarrows \cdots. \quad (5.10)$$

It provides a resolution of $F_G(F, F)$ and leads to a (conditionally convergent) Bous-field–Kan spectral sequence

$$E_r^{*,*} \Longrightarrow F_G^*(F). \quad (5.11)$$

As we started with a resolution of G as a bimodule over itself, the spectral sequence is in fact one of $F_R^*(F)$ -comodules. The unrolled exact couple underlying the spectral sequence is of the form

$$\begin{array}{ccccc} F_G^*(F) & \longleftarrow & I(F_G^*(F)) & \longleftarrow & \cdots \\ & \searrow & \nearrow & \searrow & \nearrow \\ & F_R^*(F) & & F_R^{*-1}(G \wedge F) & \end{array}$$

As the natural augmentations $F_G^*(F) \rightarrow F^*$ and $F_R^*(G \wedge^p F) \rightarrow F^*$ for $p \geq 0$ are all split and compatible, we deduce an exact couple involving reduced groups

$$\begin{array}{ccccc} \tilde{F}_G^*(F) & \longleftarrow & I(F_G^*(F)) & \longleftarrow & \cdots \\ & \searrow^{j_0} & \nearrow & \searrow^{j_1} & \nearrow \\ & \tilde{F}_R^*(F) & & \tilde{F}_R^{*-1}(G \wedge F) & \end{array} \quad (5.12)$$

We obtain a map $W_G^*: I(F_G^*(F)) \rightarrow \tilde{F}_R^{*-1}(G)$ as claimed in Proposition 5.10 by taking the composition

$$W_G^*: I(F_G^*(F)) \xrightarrow{j_1} \tilde{F}_R^{*-1}(G \wedge F) \xrightarrow{(1 \wedge \eta_F)^*} \tilde{F}_R^{*-1}(G).$$

The E_1 -term of the spectral sequence is the cochain complex which is associated to the cosimplicial R^* -module given by the homotopy groups of (5.10). As $F_R^*(G)$ is free over F^* by hypothesis, there are Künneth isomorphisms

$$E_1^{p,*} = F_R^*((G)^{\wedge p} \wedge F) \cong F_R^*(G)^{\otimes_{F^*}(p)} \otimes_{F^*} F_R^*(F).$$

The components of the normalized cochain complex are given by

$$N^{p,*} = \ker s^0 \cap \cdots \cap \ker s^p \subseteq E_1^{p,*},$$

where the s^i are the codegeneracy maps. These are induced by

$$G^{\wedge(i+1)} \wedge \eta_G \wedge G^{\wedge(p-i)} \wedge F: G^{\wedge(p+1)} \wedge F \longrightarrow G^{\wedge(p+2)} \wedge F.$$

It follows that the s^i are given by

$$a_1 \otimes \cdots \otimes a_p \otimes x \longmapsto \eta^*(a_i)a_1 \otimes \cdots \otimes \widehat{a_i} \otimes \cdots \otimes a_p \otimes x$$

where $\eta^* = F_R^*(\eta_G): F_R^*(G) \rightarrow F_R^*(R) \cong F^*$ is the augmentation and the factor under $\widehat{}$ is omitted. Hence we have

$$N^{p,*} = \widetilde{F}_R^*(G)^{\otimes_{F^*}(p)} \otimes_{F^*} F_R^*(F).$$

Now we use the fact that the spectral sequence has a multiplicative structure, which corresponds to the composition pairing on the target. It is induced by

$$F_R(G^{\wedge p} \wedge F, F) \wedge F_R(G^{\wedge q} \wedge F, F) \longrightarrow F_R(G^{\wedge(p+q)} \wedge F, F)$$

which sends $\alpha \wedge \beta$ to the composition

$$G^{\wedge(p+q)} \wedge F \xrightarrow{G^{\wedge p} \wedge \beta} G^{\wedge p} \wedge F \xrightarrow{\alpha} F.$$

The proof that this gives rise to a product structure on the spectral sequence is analogous to the one given in [9] for the homotopy spectral sequence of a space with coefficients in a ring. In particular, we find that our spectral sequence has an action of $E_1^{0,*} \cong F_R^*(F)$. It is not difficult to see that the (reduced) E_1 -term is given as an algebra as

$$T_{F^*}^{*,*}(\widetilde{F}_R^*(G)) \otimes_{F^*} F_R^*(F).$$

As the augmentation $F_R^*(F) \rightarrow F^*$ is surjective, the unit element $1 \in E_1^{0,*}$ must be a permanent cycle. By $F_R^*(F)$ -linearity, this implies that the whole of $E_1^{0,*}$ consists of permanent cycles. As a consequence, we find that $p: F_G^*(F) \rightarrow F_R^*(F)$ is surjective, as claimed in Proposition 5.9. (This argument is independent of the hypotheses made on $F_R^*(G)$ and W_B^* .) Now W_G^* is surjective by assumption, and hence $\widetilde{F}_R^*(G)$ consists of permanent cycles. The multiplicative structure now forces the whole spectral sequence to collapse, and so it converges strongly. This means that the canonical map

$$F_G^*(F) \longrightarrow \lim_s F_G^*(F)/F^s(F_G^*(F))$$

is an isomorphism, where

$$\cdots \subseteq F^{s+1}(F_G^*(F)) \subseteq F^s(F_G^*(F)) \subseteq \cdots \subseteq F^0(F_G^*(F)) = F_G^*(F)$$

is the spectral sequence filtration. As the filtration subquotients

$$F^s(\cdots)/F^{s+1}(\cdots) \cong E_\infty^{s,*} \cong \widetilde{F}_R^*(G)^{\otimes_{F^*}(s)} \otimes_{F^*} F_R^*(F)$$

are all relatively injective, there are no non-trivial extensions. Furthermore, forming extended comodules commutes with inverse limits, by definition of inverse limits of comodules (see [17, Prop. 1.2.2]). This finishes the proof of part (i).

(ii) Part (i) and Proposition 5.8 imply that

$$\mathrm{THH}_R^*(G, F) \cong \widehat{T}_{F^*}(\widetilde{F}_R^{*-1}(G)) \tag{5.13}$$

as F^* -modules. To obtain more information, we apply $F_{G \wedge G^{\text{op}}}(-, F)$ to the bar resolution (5.9). This yields a cosimplicial R -module of the form

$$\text{THH}_R(G, F) \longrightarrow R \rightrightarrows F_R(G, F) \rightrightarrows F_R(G \wedge G, F) \rightrightarrows \cdots \quad (5.14)$$

Let us consider the associated Bousfield–Kan spectral sequence

$$\bar{E}_r^{*,*} \implies \text{THH}_R^*(G, F). \quad (5.15)$$

Just as the other spectral sequence (5.11), it carries a multiplicative structure, which is compatible with the algebra structure on the target from Proposition 5.11. Using similar arguments as above, we find that the reduced E_1 -term is isomorphic as an algebra to $\text{T}_{F^*}(\tilde{F}_R^*(G))$.

Now we assert that there is a morphism of spectral sequences $\omega: \bar{E}_r^{*,*} \rightarrow E_r^{*,*}$ compatible with $\theta_G: \text{THH}_R^*(G, F) \rightarrow F_G^*(F)$ on the targets. To construct ω , we note that the resolution giving rise to (5.15) can also be obtained by applying $F_{F^e}(F \wedge_G - \wedge_G F, F)$ to (5.9). Similarly, the one inducing (5.11) is equivalent to the one obtained by applying $F_{F^e}(F \wedge_G - \wedge_G F \wedge F, F)$ to (5.9). The product map of F induces a morphism ω of the required form. On reduced E^1 -terms, ω is injective, as it is given by

$$1^{\otimes p} \otimes \nu^*: \tilde{F}_R^*(G)^{\otimes_{F^*}(p)} \longrightarrow \tilde{F}_R^*(G)^{\otimes_{F^*}(p)} \otimes_{F^*} F_R^*(F),$$

where ν^* is the augmentation of the coalgebra $F_R^*(F)$. It follows that the spectral sequence (5.15) collapses, along with (5.11). In particular, we obtain a surjection

$$\text{Der}_R^{*-1}(G, F) \cong \ker(\text{THH}_R^*(G, F) \rightarrow F^*) \longrightarrow \bar{E}_\infty^{1,*-1} = \tilde{F}_R^{*-1}(G). \quad (5.16)$$

This map coincides with the forgetful map V_G^* , as we show next. This will prove the first statement of (ii), as $F_R^*(G)$ is F^* -free by assumption and as $\text{THH}_R^*(G, F)$ and hence $\text{Der}_R^*(G, F)$ are F^* -modules by (5.13). It will also imply Proposition 5.10.

By [22, Thm. 4.1] the universal derivation $d_G: G \rightarrow D_G$ is the unique map rendering the diagram

$$\begin{array}{ccc} G & \xrightarrow{\quad} & D_G \\ \downarrow 1 \wedge \eta \wedge 1 & & \downarrow \text{can} \\ G \wedge G \wedge G & \xrightarrow{\mu \wedge 1 - 1 \wedge \mu} & G \wedge G \end{array}$$

commutative. Now the first bit of the tower which underlies the spectral sequence (5.15) is of the form

$$\begin{array}{ccccc} \cdots & \longleftarrow & \circ & \longleftarrow & D_G & \longleftarrow & \circ & \longleftarrow & G \\ & \searrow & & \nearrow e_G & & \searrow \text{can} & & \nearrow \mu & \\ & & G \wedge G \wedge G & \xrightarrow{\mu \wedge 1 - 1 \wedge \mu} & G \wedge G & & & & \end{array} \quad (5.17)$$

The map labeled e_G is uniquely determined by the requirement that it be a lift of $\text{can}: D_G \rightarrow G \wedge G$. Now $d'_G: G \wedge G \wedge G \rightarrow D_G$ obtained from d_G by extending scalars is such a lift, hence $e_G = d'_G$. By definition of V_G^* , this shows that (5.16) is V_G^* .

To prove the last statement, we choose an arbitrary section to V_G^* and consider the induced map of algebras

$$\hat{\text{T}}_{F^*}(\tilde{F}^{*-1}(G)) \longrightarrow \text{THH}_R^*(G, F).$$

It is an isomorphism on the associated graded and therefore an isomorphism itself, by [10, Thm. 2.6]. \square

6. Higher-order infinitesimal thickenings

We will now prove the following result, which easily implies Theorem 1.1 as we show below:

Theorem 6.1. *Let F be a regular quotient algebra of an even commutative \mathbb{S} -algebra R . There is a sequence of R -algebra spectra under R of the form*

$$R \longrightarrow \cdots \longrightarrow R/I^{s+1} \longrightarrow R/I^s \longrightarrow \cdots \longrightarrow R/I = F,$$

whose sequence of coefficients is the canonical sequence of R_* -algebras

$$R_* \longrightarrow \cdots \longrightarrow R_*/I^{s+1} \longrightarrow R_*/I^s \longrightarrow \cdots \longrightarrow R_*/I \cong F_*$$

and for which the homomorphisms $W_{R/I^s}^*: I(F_{R/I^s}^*(F)) \rightarrow \tilde{F}_R^{*-1}(R/I^s)$ are surjective for $s > 1$.

As $F_R^*(R/I^s)$ is F^* -free by Proposition 2.9, Theorem 5.12 then implies:

Corollary 6.2. *The forgetful map $V_{R/I^s}^*: \text{Der}_R^*(R/I^s, F) \rightarrow \tilde{F}_R^*(R/I^s)$ is split surjective. Any splitting gives rise to an algebra isomorphism*

$$\widehat{T}_F^*(\tilde{F}_R^{*-1}(R/I^s)) \cong \text{THH}_R^*(R/I^s, R/I).$$

Furthermore, there is an isomorphism of $F_R^*(F)$ -comodules

$$F_{R/I^s}^*(F) \cong \widehat{T}_{F^*}(\tilde{F}_R^{*-1}(R/I^s)) \otimes_{F^*} F_R^*(F).$$

Proof (Proof of Theorem 6.1 \implies Theorem 1.1). By definition of a regular pair, F is of the form $T \wedge L$, where $L = R/J$ is a regular quotient algebra of R . Let

$$R \longrightarrow \cdots \longrightarrow R/J^{s+1} \longrightarrow R/J^s \longrightarrow \cdots \longrightarrow R/J = L$$

be a sequence of R -algebras with the properties specified in Theorem 6.1. By applying $T \wedge -$, we obtain a sequence of R -algebras under T satisfying the required conditions (see the proof of [31, Thm. 6.6] for details). \square

Proof (Proof of Theorem 6.1). We construct the sequence inductively. The start of the induction is provided by Corollary 4.4. Assuming that the sequence is constructed up to R/I^s , we show now that $W_{R/I^s}^*: I(F_{R/I^s}^*(F)) \rightarrow \tilde{F}_R^{*-1}(R/I^s)$ is surjective. Consider the map $\pi_s: R/I^s \rightarrow F$ given by composing all the maps of the sequence constructed so far and regard it as a map in the derived category of R/I^s -bimodules. Let

$$I/I^s \xrightarrow{\iota_s} R/I^s \xrightarrow{\pi_s} F \xrightarrow{\tau_s} \Sigma I/I^s \quad (6.1)$$

be a cofibre sequence associated to π_s and let

$$\Sigma^{-1} R/I^s \longrightarrow D_{R/I^s} \longrightarrow R/I^s \wedge R/I^s \xrightarrow{\mu_s} R/I^s \quad (6.2)$$

be a cofibre sequence associated to the product μ_s of R/I^s . Smashing the latter over R/I^s with F and R/I^s from the right respectively gives rise to a diagram (we write \wedge_s for \wedge_{R/I^s} and D_s for D_{R/I^s})

$$\begin{array}{ccccccc} \Sigma^{-1} F & \longrightarrow & D_s \wedge_s F & \longrightarrow & R/I^s \wedge F & \longrightarrow & F \\ & & \uparrow 1 \wedge \pi_s & & \uparrow 1 \wedge \pi_s & & \\ \Sigma^{-1} R/I^s & \longrightarrow & D_s & \longrightarrow & R/I^s \wedge R/I^s & \longrightarrow & R/I^s \end{array}$$

By the 3×3 -Lemma [19, Lemma A.1.2], we can complete it to one of the form

$$\begin{array}{ccccccc} \Sigma^{-1} X_s & \longrightarrow & D_s \wedge_s \Sigma I/I^s & \longrightarrow & R/I^s \wedge \Sigma I/I^s & \longrightarrow & X_s \\ \uparrow & & \uparrow 1 \wedge \tau_s & & \uparrow 1 \wedge \tau_s & & \uparrow \\ \Sigma^{-1} F & \longrightarrow & D_s \wedge_s F & \longrightarrow & R/I^s \wedge F & \longrightarrow & F \\ \uparrow & & \uparrow 1 \wedge \pi_s & & \uparrow 1 \wedge \pi_s & & \uparrow \\ \Sigma^{-1} R/I^s & \longrightarrow & D_s & \longrightarrow & R/I^s \wedge R/I^s & \longrightarrow & R/I^s \\ \uparrow & & \uparrow 1 \wedge \iota_s & & \uparrow 1 \wedge \iota_s & & \uparrow \\ \Sigma^{-2} X_s & \longrightarrow & D_s \wedge_s I/I^s & \longrightarrow & R/I^s \wedge I/I^s & \longrightarrow & \Sigma^{-1} X_s, \end{array} \quad (6.3)$$

for some X_s , such that all squares commute, except for the bottom left one, which anti-commutes, and such that all the rows and columns are cofibre sequences. We apply $F_s^*(-) = F_{R/I^s}^*(-)$ and obtain

$$\begin{array}{ccccccc}
 & \cdots & & \cdots & & \cdots & & \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longleftarrow F_s^{*+1}(X_s) & \longleftarrow F_s^*(D_s \wedge_s \Sigma I/I^s) & \longleftarrow F_R^{*-1}(I/I^s) & \longleftarrow F_s^*(X_s) & \longleftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longleftarrow F_s^{*+1}(F) & \longleftarrow F_s^*(D_s \wedge_s F) & \longleftarrow F_R^*(F) & \longleftarrow F_s^*(F) & \longleftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longleftarrow F^{*+1} & \longleftarrow F_s^*(D_s) & \longleftarrow F_R^*(R/I^s) & \longleftarrow F^* & \longleftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 \cdots & \longleftarrow F_s^{*+2}(X_s) & \longleftarrow F_s^*(D_s \wedge_s I/I^s) & \longleftarrow F^*(I/I^s) & \longleftarrow F_s^{*+1}(X_s) & \longleftarrow \cdots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & \cdots & & \cdots & & \cdots & & \cdots
 \end{array}$$

(*)

The fact that square (*) commutes implies that the vertical map $R/I^s \rightarrow F$ in the right-most column of (6.3) is the map π_s from above, when viewed as a left R/I^s -linear map. Therefore this column is equivalent to the cofibre sequence (6.1) of left R/I^s -modules. Now observe that the map canonical $F_s^*(F) \rightarrow F_R^*(F)$ is surjective, because it factors through the surjection $F_{R/I^2}^*(F) \rightarrow F_R^*(F)$ (Corollary 4.7). We claim that $F_s^{*+1}(X_s) \cong F_s^*(I/I^s) \rightarrow F_R^*(I/I^s)$ is surjective, too. To show this, we “invert” the sequence $R/I^s \rightarrow \cdots \rightarrow R/I^2 \rightarrow F$ (of R/I^s -bimodules) by repeated use of the octahedral axiom to obtain a sequence

$$I/I^s \longrightarrow I/I^{s-1} \longrightarrow \cdots \longrightarrow I/I^3 \longrightarrow I/I^2.$$

We show inductively that the natural maps $F_s^*(I/I^t) \rightarrow F_R^*(I/I^t)$ are surjections for $t \leq s$. This is the case for $t = 2$, as I/I^2 is a wedge of suspension of F . Assume that $F_s^*(I/I^t) \rightarrow F_R^*(I/I^t)$ is surjective. Note that the homotopy fibre of $I/I^{t+1} \rightarrow I/I^t$ is equivalent to the one of $R/I^{t+1} \rightarrow R/I^t$, which is I^t/I^{t+1} by construction. Consider the exact sequences

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & F_s^*(I/I^t) & \longrightarrow & F_s^*(I/I^{t+1}) & \longrightarrow & F_s^*(I^t/I^{t+1}) & \longrightarrow & F_s^{*+1}(I/I^t) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & F_R^*(I/I^t) & \longrightarrow & F_R^*(I/I^{t+1}) & \longrightarrow & F_R^*(I^t/I^{t+1}) & \longrightarrow & F_R^{*+1}(I/I^t) & \longrightarrow & \cdots
 \end{array}$$

As I^t/I^{t+1} is given as a wedge of suspensions of F , the inductive step follows from the 5-Lemma. What we have seen implies, together with Proposition 2.9, that we can extract from (6.3) a diagram with exact rows and sequences of the form

$$\begin{array}{ccccccc}
 & 0 & & & & & \\
 & \downarrow & & & & & \\
 & \tilde{F}_R^{*-1}(R/I^s) & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
 0 & \longleftarrow F_R^{*-1}(I/I^s) & \longleftarrow F_s^*(X_s) & \longleftarrow F^{*-1}(D_s \wedge_s \Sigma I/I^s) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longleftarrow \tilde{F}_R^*(F) & \longleftarrow \tilde{F}_s^*(F) & \longleftarrow F_s^{*-1}(D_s \wedge_s F) & \longleftarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow (1 \wedge \pi_s)^* & \\
 & 0 & & 0 & & F_s^{*-1}(D_s) &
 \end{array}$$

The snake lemma shows that the upper vertical map in the right column is injective, the lower one, $(1 \wedge \pi_s)^*$, is surjective and that $F_s^*(D_s) \cong \tilde{F}_R^*(R/I^s)$. We show now that we can factorize $W_{R/I^s}^* : I(F_s^*(F)) \rightarrow \tilde{F}_R^{*-1}(R/I^s)$ as

$$I(F_s^*(F)) \cong F_s^{*-1}(D_s \wedge_s F) \xrightarrow{(1 \wedge \pi_s)^*} F_s^{*-1}(D_s) \xrightarrow{\psi} \tilde{F}_R^{*-1}(R/I^s), \quad (6.4)$$

where ψ is an isomorphism. This will imply surjectivity of W_{R/I^s}^* .

Recall the definition of W_{R/I^s}^* , based on the exact couple (5.12) (for $G = R/I^s$). This exact couple can be obtained by applying $F_s^*(-)$ to a diagram of the form

$$\begin{array}{ccccc} F & \xrightarrow{\circ} & D_s \wedge_s F & \xrightarrow{\circ} & \dots \\ & \nwarrow & \swarrow & \nwarrow & \swarrow \\ & R/I^s \wedge F & & R/I^s \wedge R/I^s \wedge F & \end{array}$$

φ

and taking reduced cohomology where appropriate. Arguing as after for (5.17), we find that φ must be the map obtained from the universal derivation $d_s : R/I^s \rightarrow D_s$ by first extending scalars and then applying $- \wedge_s F$. Therefore W_{R/I^s}^* is given as the composition

$$I(F_s^*(F)) \cong F_s^{*-1}(D_s \wedge_s F) \xrightarrow{\varphi^*} \tilde{F}_s^{*-1}(R/I^s \wedge R/I^s \wedge F) \xrightarrow{(1 \wedge 1 \wedge \eta_F)^*} \tilde{F}_s^{*-1}(R/I^s \wedge R/I^s) \cong \tilde{F}_R^{*-1}(R/I^s).$$

Denoting by $\varphi' : R/I^s \wedge R/I^s \rightarrow D_s$ the map induced from d_s by extending scalars to R/I^s on the left, we have a commutative diagram

$$\begin{array}{ccc} R/I^s \wedge R/I^s & \xrightarrow{\varphi'} & D_s \\ \downarrow 1 \wedge 1 \wedge \eta_F & & \downarrow 1 \wedge \pi_s \\ R/I^s \wedge R/I^s \wedge F & \xrightarrow{\varphi} & D_s \wedge_s F. \end{array}$$

It allows us to factor W_{R/I^s}^* as

$$I(F_s^*(F)) \cong F_s^{*-1}(D_s \wedge_s F) \xrightarrow{(1 \wedge \pi_s)^*} \tilde{F}_s^{*-1}(D_s) \xrightarrow{(\varphi')^*} \tilde{F}_s^{*-1}(R/I^s \wedge R/I^s) \cong \tilde{F}_R^{*-1}(R/I^s).$$

It is straightforward to check that the map $D_s \rightarrow R/I^s \wedge R/I^s$ from the fibre sequence (6.2) induces an inverse to $(\varphi')^*$. Therefore, we have found a factorization of W_{R/I^s}^* of the form (6.4) and shown that it is surjective.

We now construct R/I^{s+1} as an R -algebra. Theorem 5.12 implies that

$$V_{R/I^s}^* : \text{Der}_R^*(R/I^s, \Sigma I^s/I^{s+1}) \rightarrow (\widetilde{\Sigma I^s/I^{s+1}})^*(R/I^s)$$

is surjective, as I^s/I^{s+1} is a wedge of suspensions of F . Observe that the map $\theta^s : R/I^s \rightarrow \Sigma I^s/I^{s+1}$ from the tower (2.4) is contained in the reduced cohomology group

$$(\widetilde{I^s/I^{s+1}})_R^*(R/I^s) = \ker((I^s/I^{s+1})_R^*(R/I^s) \xrightarrow{\eta_{R/I^s}^*} (I^s/I^{s+1})_R^*(R) \cong I^s/I^{s+1}).$$

This is clear, as $R/I^{s+1} \xrightarrow{\rho} R/I^s \xrightarrow{\theta^s} \Sigma I^s/I^{s+1}$ is zero and as the unit η_{R/I^s} factors through ρ . Thus we can realize θ_s as a strict derivation and therefore construct R/I^{s+1} as the associated singular extension. \square

Remark 6.3. If R is the Eilenberg–MacLane spectrum $H\Lambda$ of a commutative ring Λ , Corollary 6.2 states that

$$\text{Ext}_{\Lambda/I^s}^*(\Lambda/I, \Lambda/I) \cong T_{\Lambda/I}^*(\widetilde{\text{Ext}}_{\Lambda}^{*-1}(\Lambda/I^s, \Lambda/I)) \otimes_{\Lambda} \text{Ext}_{\Lambda}^*(\Lambda/I, \Lambda/I), \quad (6.5)$$

by [13, IV.2]. This is a well-known fact in commutative algebra. It is usually considered in the case where Λ is a regular local ring with maximal ideal I . Eq. (6.5) implies that Λ/I^s is a Golod ring under this assumption. See [3] for details.

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References

- [1] V. Angeltveit, A_∞ -obstruction theory and the strict associativity of E/I . <http://www.math.uchicago.edu/~vigleik/>, Preprint.
- [2] V. Angeltveit, Topological Hochschild homology and cohomology of A_∞ ring spectra. [arXiv:math.AT/0612164](https://arxiv.org/abs/math/0612164), Preprint.
- [3] L.L. Avramov, Small homomorphisms of local rings, *J. Algebra* 50 (2) (1978) 400–453.
- [4] A. Baker, A_∞ structures on some spectra related to Morava K -theories, *Quart. J. Math. Oxford Ser. (2)* 42 (2) (1991) 403–419.
- [5] A. Baker, A. Jeanneret, Brave new Hopf algebroids and extensions of MU -algebras, *Homology Homotopy Appl.* 4 (1) (2002) 163–173 (electronic).
- [6] A. Baker, A. Lazarev, On the Adams spectral sequence for R -modules, *Algebr. Geom. Topol.* 1 (2001) 173–199.
- [7] A. Baker, Birgit Richter, On the Γ -cohomology of rings of numerical polynomials and E_∞ structures on K -theory, *Comment. Math. Helv.* 80 (2005) 691–723.
- [8] M. Bastera, André-Quillen cohomology of commutative S -algebras, *J. Pure Appl. Algebra* 144 (1999) 111–143.
- [9] A.K. Bousfield, D.M. Kan, Pairings and products in the homotopy spectral sequence, *Trans. Amer. Math. Soc.* 177 (1973) 319–343.
- [10] J.M. Boardman, Conditionally convergent spectral sequences, in: *Homotopy Invariant Algebraic Structures* (Baltimore, MD, 1998), in: *Contemp. Math.*, vol. 239, Amer. Math. Soc., Providence, RI, 1999, pp. 49–84.
- [11] W.G. Dwyer, P.S. Hirschhorn, D.M. Kan, J.H. Smith, Homotopy limit functors on model categories and homotopical categories, in: *Mathematical Surveys and Monographs*, vol. 113, American Mathematical Society, Providence, RI, 2004.
- [12] S. Eilenberg, J.C. Moore, Foundations of relative homological algebra, *Mem. Amer. Math. Soc.* 55 (1965) 39.
- [13] A.D. Elmendorf, I. Kriz, M.A. Mandell, J.P. May, Rings, modules, and algebras in stable homotopy theory, in: *Mathematical Surveys and Monographs*, vol. 47, American Mathematical Society, Providence, RI, 1997, With an Appendix by M. Cole.
- [14] P. Goerss, H.-W. Henn, M. Mahowald, C. Rezk, A resolution of the $K(2)$ -local sphere at the prime 3, *Ann. of Math. (2)* 162 (2005) 777–822.
- [15] P.G. Goerss, M.J. Hopkins, Moduli spaces of commutative ring spectra, in: *Structured Ring Spectra*, in: *London Math. Soc. Lecture Note Ser.*, vol. 315, Cambridge Univ. Press, Cambridge, 2004, pp. 151–200.
- [16] R. Hartshorne, Algebraic geometry, in: *Graduate Texts in Mathematics*, vol. 52, Springer-Verlag, New York, 1977.
- [17] M. Hovey, Homotopy theory of comodules over a Hopf algebroid, *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory*, in: *Contemp. Math.*, vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 261–304.
- [18] M. Hovey, Model categories, in: *Mathematical Surveys and Monographs*, vol. 63, American Mathematical Society, Providence, RI, 1999.
- [19] M. Hovey, J.H. Palmieri, N.P. Strickland, Axiomatic stable homotopy theory, *Mem. Amer. Math. Soc.* 128 (1997) x+114.
- [20] M. Hovey, B. Shipley, J. Smith, Symmetric spectra, *J. Amer. Math. Soc.* 13 (2000) 149–208.
- [21] L. Illusie, Grothendieck’s existence theorem in formal geometry, in: *Fundamental Algebraic Geometry*, *Amer. Math. Soc.* 123 (2005) 179–233. With a letter (in French) of Jean-Pierre Serre.
- [22] A. Lazarev, Homotopy theory of A_∞ ring spectra and applications to MU -modules, *K-Theory* 24 (2001) 243–281.
- [23] A. Lazarev, Towers of MU -algebras and the generalized Hopkins–Miller theorem, *Proc. London Math. Soc. (3)* 87 (2003) 498–522.
- [24] H.R. Miller, On relations between Adams spectral sequences, with an application to the stable homotopy of a Moore space, *J. Pure Appl. Algebra* 20 (1981) 287–312.
- [25] D.C. Ravenel, Complex cobordism and stable homotopy groups of spheres, in: *Pure and Applied Mathematics*, vol. 121, Academic Press Inc., Orlando, FL, 1986.
- [26] B. Richter, A. Robinson, Gamma homology of group algebras and of polynomial algebras, in: *Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-Theory*, in: *Contemp. Math.*, vol. 346, Amer. Math. Soc., Providence, RI, 2004, pp. 453–461.
- [27] A. Robinson, Obstruction theory and the strict associativity of Morava K -theories, in: *Advances in Homotopy Theory* (Cortona, 1988), in: *London Math. Soc. Lecture Note Ser.*, vol. 139, Cambridge Univ. Press, Cambridge, 1989, pp. 143–152.
- [28] J. Rognes, Galois extensions of structured ring spectra, *Mem. Amer. Math. Soc.* (2007) (in press).
- [29] S. Schwede, B.E. Shipley, Algebras and modules in monoidal model categories, *Proc. London Math. Soc. (3)* 80 (2000) 491–511.
- [30] N.P. Strickland, Products on MU -modules, *Trans. Amer. Math. Soc.* 351 (1999) 2569–2606.
- [31] S. Wüthrich, I -adic towers in topology, *Algebr. Geom. Topol.* 5 (2005) 1589–1635 (electronic).